Some "official solutions"

Your name: ______________________

Your TA's name: ______________________

Your section number and/or day and time: __________

Math 191, Prelim 1

Thursday, Sept 27th, 2007. 7:30 – 9:00 PM

This exam should have 8 pages, with 5 problems adding up to 100 points.

The last two pages are blank and can be used as scrap paper for computations and checking answers.

No calculators or books allowed - You may have one 8.5x11 formula sheet.

To improve your chances of getting full credit (or maximum partial credit) and to ease the work of the graders, please:

- write clearly and legibly;
- box in your answers;
- simplify your answers as much as possible;
- explain your answers as completely as time and space allow.

Problem 1:	/20
Problem 2:	/20
Problem 3:	/20
Problem 4:	/20
Problem 5:	/20
TOTAL:	/100

Academic Integrity is expected of all students of Cornell University at all times, whether in the presence or absence of members of the faculty.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

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Signature of the Student

[20] 1. Evaluate the following:

(a) $\displaystyle\int_{-\pi/3}^{\pi/3} x^2 \sin x^3\, dx$ $= 0$ odd function.

OR

$x^3 = u$; $3x^2 dx = du$ $\displaystyle\frac{1}{3}\int_{-(\frac{\pi}{3})^3}^{+(\frac{\pi}{3})^3} \sin u\, du = -\frac{1}{3} \cos u \Big|_{-\frac{\pi^3}{27}}^{\frac{\pi^3}{27}}$

$= -\frac{1}{3}\left(\cos\frac{\pi^3}{27} - \cos\left(-\frac{\pi^3}{27}\right)\right) = 0$

(b) $\displaystyle\int \frac{1}{x^2}\sqrt{1+\frac{1}{x}}dx$ $1+\frac{1}{x} = u$ $-\frac{1}{x^2}dx = du$.

$\parallel$

$-\int \sqrt{u}\, du = \dfrac{-u^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{2}{3}\left(1+\frac{1}{x}\right)^{\frac{3}{2}} + C$

(c) $\displaystyle\frac{d}{dx}\int_{x-1}^{\cos x} \sqrt{1-t^2}dt = ?$ $= \displaystyle\frac{d}{dx}\int_0^{\cos x}\sqrt{1-t^2}dt - \frac{d}{dx}\int_0^{x-1}\sqrt{1-t^2}\,dt$

$= \sqrt{1-\cos^2 x}\cdot \frac{d}{dx}(\cos x) - \sqrt{1-(x-1)^2}\,\frac{d(x-1)}{dx}$

$= -\sin x\ |\sin x| - \sqrt{-x^2+2x}$.

(d) If $f(t)$ is differentiable and $x\sin x = \displaystyle\frac{d}{dx}\int_0^{x^2} f(t)dt$, $f(x) = ?$

$x\sin x = \displaystyle\frac{d}{dx}\int_0^{x^2} f(t)\,dt = f(x^2)\,\frac{d\,x^2}{dx} = f(x^2)\,2x$.

$\Rightarrow f(x^2) = \dfrac{\sin x}{2}$

$\Rightarrow f(x) = \dfrac{\sin\sqrt{x}}{2}$.

(NOTE that

$\cdot)$ $\dfrac{\sin(-\sqrt{x})}{2}$ is not a solution

$\cdot)$ $\dfrac{\sin\sqrt{x}}{2} + C$ is not a solution.

$* \cos^2 t + \sin^2 t = 1$

[20] 2. Consider a parametric function defined by $x(t) = r(t)\cos t$ and $y(t) = r(t)\sin t$ for $0 \le t \le 2\pi$.

(a) In terms of $r(t)$ and its derivative(s) find an expression for the arc length of the curve .

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = r'(t)\cos t - r(t)\sin t$$

$$\frac{dy}{dt} = r'(t)\sin t + r(t)\cos t$$

$$\boxed{L = \int_0^{2\pi} \sqrt{r'(t)^2 + r(t)^2}\, dt}$$

$$(dx/dt)^2 + \left(\frac{dy}{dt}\right)^2 = r'(t)^2\cos^2 t - 2r'(t)r(t)\cos t\sin t$$
$$+ r(t)^2\sin^2 t + r'(t)^2\sin^2 t + 2r'(t)(r(t))\cos t\sin t$$
$$+ r(t)^2\cos^2 t$$
$$= r'(t)^2(\cos^2 t + \sin^2 t) + r(t)^2(\cos^2 t + \sin^2 t$$
$$= r'(t)^2 + r(t)^2$$

(b) A cardioid is a parametric plot that looks like a heart. It is defined by

$$x(t) = (1 - \cos t)\cos t$$
$$y(t) = (1 - \cos t)\sin t$$

for $0 \le t \le 2\pi$. Find its arc length.

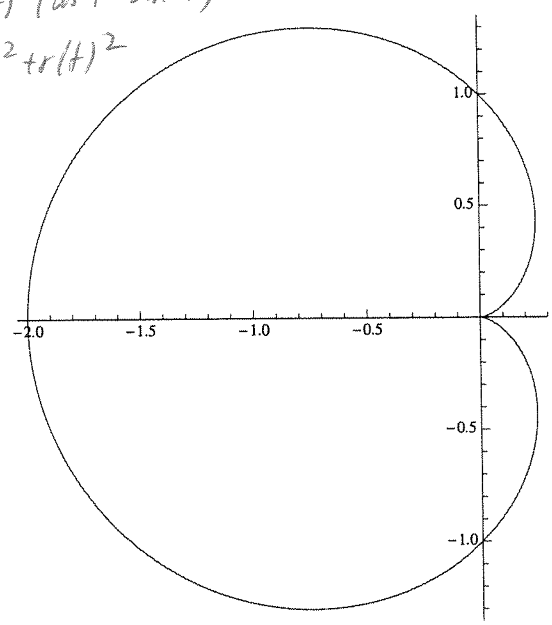


Figure 1: Cardioid

$$r(t) = 1 - \cos t$$
$$r'(t) = \sin t$$

$$\text{so } L = \int_0^{2\pi} \sqrt{\sin^2 t + (1-\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + 1 - 2\cos t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\left(\frac{1-\cos t}{2}\right)}\, dt$$

$$= 2\int_0^{2\pi} \sqrt{\frac{1-\cos t}{2}}\, dt$$

$$= 2\int_0^{2\pi} \sqrt{\sin^2 t/2}\, dt$$

$$= 2\int_0^{2\pi} \sin\tfrac{t}{2}\, dt$$

$$= 2\left(-2\cos\tfrac{t}{2}\right)\Big]_0^{2\pi}$$

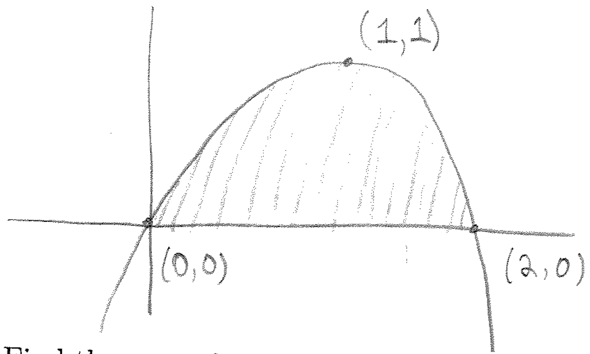
$$= -4\left(\cos\tfrac{2\pi}{2} - \cos\tfrac{0}{2}\right)$$

$$= -4(\cos\pi - \cos 0)$$

$$= -4(-1 - -1) = -4 \cdot -2 = \boxed{8}$$

[20] 3. For the region enclosed by $y = 2x - x^2$ and the x-axis evaluate the following:

(a) Sketch the region.



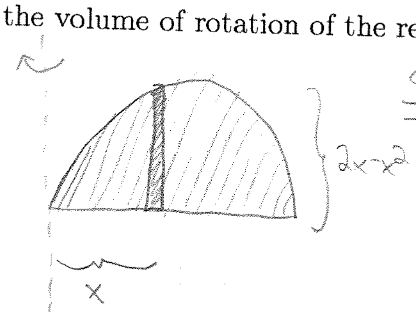
(b) Find the area of the region.

$$A = \int_0^2 (2x - x^2)\,dx = x^2 - \frac{x^3}{3}\Big|_{x=0}^{x=2}$$

$$= \left(4 - \frac{8}{3}\right) - (0 - 0)$$

$$= \boxed{\frac{4}{3}}$$

(c) Find the volume of rotation of the region about the line $x = 0$ (y-axis).



Shell method:

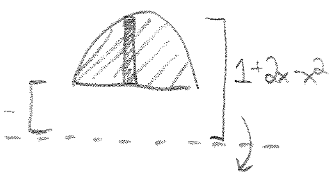
$$V = \int_0^2 2\pi x (2x - x^2)\,dx$$

$$= \int_0^2 2\pi (2x^2 - x^3)\,dx$$

$$= 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_{x=0}^{x=2}$$

$$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \boxed{\frac{8\pi}{3}}$$

(d) Find the volume of rotation of the region about the line $y = -1$.



Washer method:

$$V = \pi \int_0^2 (1 + 2x - x^2)^2 - 1^2\,dx$$

$$= \pi \int_0^2 1 + 2x - x^2 + 2x + 4x^2 - 2x^3 - x^2 - 2x^3 + x^4 - 1\,dx$$

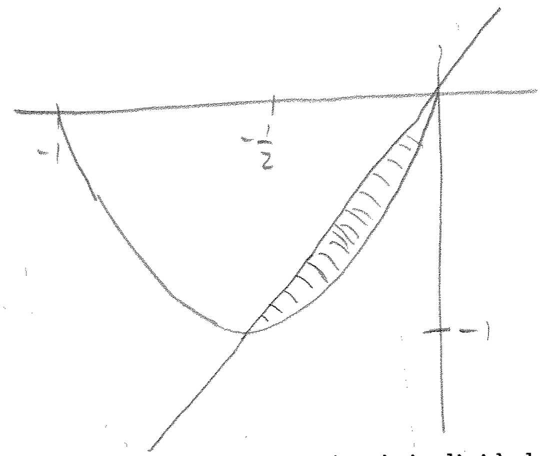
$$= \pi \int_0^2 x^4 - 4x^3 + 2x^2 + 4x\,dx$$

$$= \pi \left(\frac{x^5}{5} - x^4 + \frac{2}{3}x^3 + 2x^2\right)\Big|_{x=0}^{x=2}$$

$$= \pi\left(\frac{32}{5} - 16 + \frac{16}{3} + 8\right) = \boxed{\frac{56}{15}\pi}$$

[20] 4. Find the area between the curves $f(x) = 4x^2 + 4x$ and $g(x) = 2x$.

(a) Sketch the curves and shade the desired area.



(5 pts.)

(b) Set up the Riemann sum for the area, assuming it is divided into n intervals of equal width.

$\Delta x = \frac{0-(-\frac{1}{2})}{n} = \frac{1}{2n}$ (2 pts.)

$c_k = \frac{-k}{2n}$, $k=1,\dots,n$ (3 pts.)

$\sum_{k=1}^{n} [g(c_k) - f(x_k)]\Delta x = \sum_{k=1}^{n} \frac{1}{2n}\left[-4\left(\frac{-k}{2n}\right)^2 - 2\left(\frac{-k}{2n}\right)\right]$ (5 pts.)

(c) Evaluate the Riemann sum as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} [g(c_k) - f(c_k)]\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n}\left[\frac{-k^2}{n^2} + \frac{k}{n}\right]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{-k^2}{2n^3} + \sum_{k=1}^{n} \frac{k}{2n^2}$$

$$= \lim_{n\to\infty} \frac{-n(n+1)(2n+1)}{12n^3} + \frac{n(n+1)}{4n^2}$$

$$= \lim_{n\to\infty} \frac{-2n^3 - 3n^2 - n}{12n^3} + \frac{n^2+n}{4n^2}$$

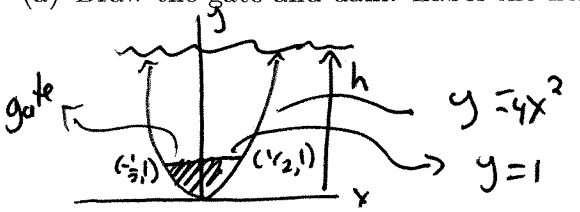
$$= -\frac{1}{6} + \frac{1}{4} = \boxed{\frac{1}{12}}$$

(5 pts.)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

[20] 5. A dam is built on a flood-prone river. At the bottom of the dam is an emergency release gate shaped like a parabola with a flat top (enclosed by the curves $y = 4x^2$ and $y = 1$). If the force on the gate exceeds 110kN, the gate will open and spill water until the force has dropped to 100kN. The goal of this problem is to find the water level when the gate first opens and when it closes. For w use the following: the density of water is 1000kg/m^3 and the acceleration due to gravity is g=10m/s^2

(a) Draw the gate and dam. Label the height of the water above the bottom of the gate.



(b) Derive an expression to describe the force on a thin strip of the gate ΔA.

$$\Delta F = (\text{weight density})\cdot(\text{distance to surface})\cdot\Delta A$$

$$\Delta F = \rho g\,(h-y)\,\Delta A \qquad \text{Note: } \rho g = w$$

$$\boxed{\Delta F = \rho g\,(h-y)\Delta A} \qquad \Delta A = L(y)\,\Delta y = 2x\,\Delta y = \sqrt{y}\,\Delta y$$

(c) Use the answer in (b) to find the total force on the gate due to water pressure.

$$F = \int dF = \int_0^1 \rho g\,(h-y)\sqrt{y}\,dy = \int_0^1 \rho g h\, y^{1/2}\,dy - \int_0^1 \rho g\, y^{3/2}\,dy$$

$$= \rho g h\left[2/3\right] - \rho g\left[2/5\right]$$

$$\boxed{F = \rho g\left[\tfrac{2}{3}h - \tfrac{2}{5}\right]}$$

(d) Solve your expression for h, the height of the water above the base of the gate.

$$F = \rho g\left[\tfrac{2}{3}h - \tfrac{2}{5}\right]$$

$$\frac{F}{\rho g} + \frac{2}{5} = \frac{2}{3}h \longrightarrow \boxed{h = \frac{3}{2}\left[\frac{F}{\rho g} + \frac{2}{5}\right]}$$

(e) Find the water level when the gate first opens.

$$h = \frac{3}{2}\left[\frac{110\cdot 1000}{10\cdot 1000} + \frac{2}{5}\right] = \boxed{\frac{3}{2}\left[11 + \frac{2}{5}\right]\text{m}}$$

(f) Find the water level when the gate closes.

$$h = \frac{3}{2}\left[\frac{100\cdot 1000}{10\cdot 1000} + \frac{2}{5}\right] = \boxed{\frac{3}{2}\left[10 + \frac{2}{5}\right]\text{m}}$$