Your name: ______________________________

Your TA's name: ______________________________

Your section number and/or day and time: ____________

Math 191, Prelim 2

Thursday, Sept 27th, 2007. 7:30 – 9:00 PM

This exam should have 8 pages, with 5 problems adding up to 100 points.

The last two pages are blank and can be used as scrap paper for computations and checking answers.

No calculators or books allowed - You may have one 8.5x11 formula sheet.

To improve your chances of getting full credit (or maximum partial credit) and to ease the work of the graders, please:

- write clearly and legibly;
- box in your answers;
- simplify your answers as much as possible;
- explain your answers as completely as time and space allow.

Problem 1:	/20
Problem 2:	/15
Problem 3:	/25
Problem 4:	/15
Problem 5:	/25
TOTAL:	**/100**

Academic Integrity is expected of all students of Cornell University at all times, whether in the presence or absence of members of the faculty.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

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Signature of the Student

[20] 1. Evaluate the following:

(a) $\displaystyle\int \frac{e^{2x}}{3+e^{2x}}dx$

Solution. Substitute $u = 3 + e^{2x}$, which means $du = 2e^{2x}\,dx$.

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(3+e^{2x}) + C = \boxed{\ln\sqrt{3+e^{2x}} + C}$$

(We can drop the absolute value because $3 + e^{2x}$ is always positive. The last step, of moving the $1/2$ to an exponent, is entirely optional.)

(b) $\displaystyle\int \frac{\cos^3 x}{\sin^3 x - \sin x}dx$

Solution.

$$\int \frac{\cos^3 x}{\sin^3 x - \sin x}dx = \int \frac{\cos^3 x}{\sin x(\sin^2 x - 1)}dx$$

Use the identitiy $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}\int \frac{\cos^3 x}{\sin^3 x - \sin x}dx &= -\int \frac{\cos^3 x}{\sin x \cos^2 x}dx \\ &= -\int \frac{\cos x}{\sin x}dx\end{aligned}$$

Substitute $u = \sin x$ giving $du = \cos x dx$.

$$\begin{aligned}\int \frac{\cos^3 x}{\sin^3 x - \sin x}dx &= -\int \frac{1}{u}du \\ &= -\ln|u| \\ &= \boxed{-\ln|\sin x| + C}\end{aligned}$$

(c) $\displaystyle\int \frac{dx}{e^{2x}-1}$

Solution. Substitute $u = e^x$, $du = e^x\,dx$, and use partial fractions:

$$\int \frac{dx}{e^x - 1} = \int \frac{du}{u(u-1)} = \int \left(\frac{1}{u-1} - \frac{1}{u} \right) du =$$

$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{e^x - 1}{e^x}\right| + C = \ln|1 - e^{-x}| + C\ .$$

[20] 2. Answer the following questions:

(a) Take the derivative of $(1+5^x)e^{\sin x}$ with respect to x.

Solution. We know that

$$5^x = e^{\ln 5^x} = e^{x \ln 5}$$

Then

$$\begin{aligned}\frac{d}{dx}(1+5^x)e^{\sin x} &= \frac{d}{dx}(e^{\sin x} + e^{x\ln 5}e^{\sin x}) \\ &= e^{\sin x}\cos x + \ln 5 e^{x\ln 5}e^{\sin x} + e^{x\ln 5}e^{\sin x}\cos x \\ &= (\cos x + \ln 5 e^{x\ln 5} + e^{x\ln 5}\cos x)e^{\sin x} \\ &= \boxed{(\cos x + 5^x \ln 5 + 5^x \cos x)e^{\sin x}}\end{aligned}$$

(b) Evaluate $\cos\left(\csc^{-1}\frac{\sqrt{x^2+9}}{x}\right)$.

Solution. Let $\theta = \csc^{-1}\dfrac{\sqrt{x^2+9}}{x}$. Then $\csc\theta = \dfrac{\sqrt{x^2+9}}{x}$ or $\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$. Draw a reference triangle with θ in one of the acute angles. Label the hypotenuse as $\sqrt{x^2+9}$, the opposite leg as x, and the adjacent leg as 3. Then from the picture it is easy to see that

$$\cos\theta = \frac{3}{\sqrt{x^2+9}}$$

(c) Find the area bounded by the curves $y = x\log_a x$, $x = e$, and the x-axis.

Solution. Use integration by parts. Let $u = \log_a x$ and let $dv = xdx$ since we can easily integrate

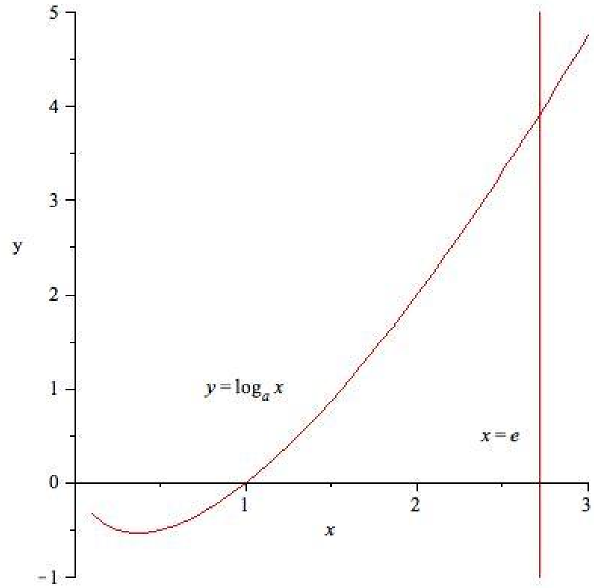


Figure 1: Plot of curves.

it. Since $\log_a x = \frac{\ln x}{\ln a}$ we get $du = \frac{1}{x\ln a}dx$. Also $v = \frac{x^2}{2}$.

$$\begin{aligned}
\int_1^e x\log_a x dx &= \left[\frac{x^2}{2}\log_a x\right]_1^e - \frac{1}{2\ln a}\int_1^e xdx \\
&= \left[\frac{x^2 \ln x}{2\ln a}\right]_1^e - \frac{1}{2\ln a}\left[\frac{x^2}{2}\right]_1^e \\
&= \frac{e^2}{2\ln a} - \frac{e^2}{4\ln a} + \frac{1}{4\ln a} \\
&= \boxed{\frac{1}{4\ln a}(e^2+1)}
\end{aligned}$$

[25] 3. Answer the following parts concerning $\operatorname{sech} x$ and $\operatorname{sech}^{-1} x$.

(a) Graph the function $f(x) = \operatorname{sech} x$ and find its domain and range.

Solution. We write $f(x)$ explicitly as

$$f(x) = \operatorname{sech} x = \frac{2}{e^x + e^{-x}},$$

and note that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$ and $f(0) = 1$. The graph of $\operatorname{sech} x$ is given by

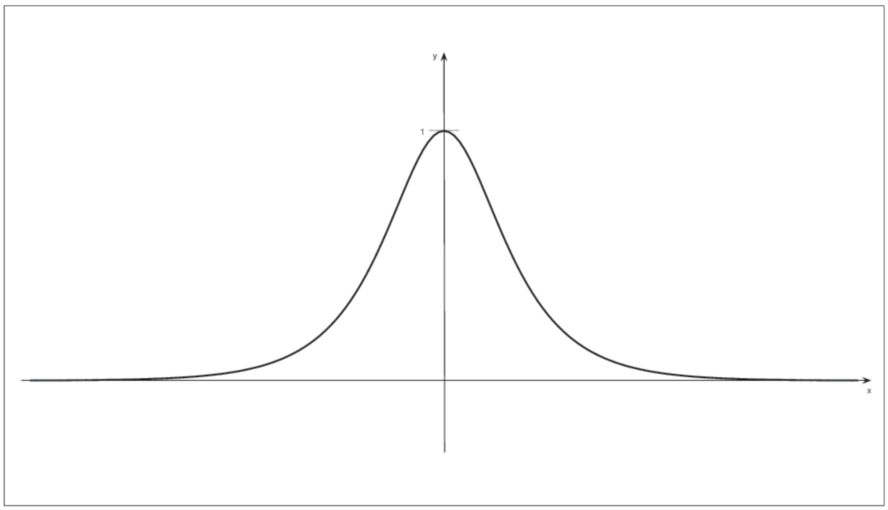


Figure 2: $\operatorname{sech} x$

The domain is $\mathbf{R}$ (the whole real line) and the range is $(0, 1]$.

(b) Does $f(x)$ have an inverse on this domain? If not, find an interval, as large as possible, on which $f(x)$ is invertible.

Solution. The function is not one to one (as can be seen from the graph) and, therefore, not invertible. We can make it one to one by taking the nonnegative half of the domain, $x \geq 0$ (the nonpositive part of the x-axis would be just as good).

(c) Using f or its restriction to the domain on which it is invertible (depending on your answer to part (b)), graph $f^{-1}(x) = \operatorname{sech}^{-1} x$ and give its domain and range.

Solution. We only consider the right half of the curve and invert it:

The modified domain for $f^{-1}(x)$ is $(0, 1]$ and the range is $[0, \infty)$.

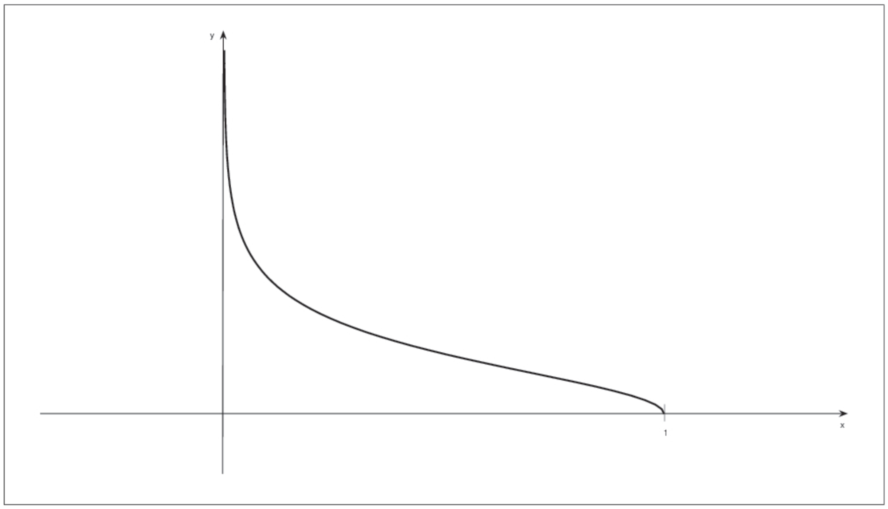


Figure 3: $\operatorname{sech}^{-1} x$

(d) Using the definition of $\operatorname{sech} x$ (in terms of $\cosh x$ or exponential functions), find $f'(x) = \frac{d}{dx} \operatorname{sech} x$.
Solution.

$$f'(x) = \frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} \frac{1}{\cosh x} = \frac{-1}{\cosh^2 x} \cdot \sinh x = -\tanh x \operatorname{sech} x.$$

(e) Find $(f^{-1}(x))' = \frac{d}{dx} \operatorname{sech}^{-1} x$.
Hint: Use part d. You may also use the identity $\tanh^2 x = 1 - \operatorname{sech}^2 x$.
Solution. Using the inverse function Theorem we have

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\tanh(\operatorname{sech}^{-1} x) \cdot \operatorname{sech}(\operatorname{sech}^{-1} x)} = \frac{-1}{x \cdot \tanh(\operatorname{sech}^{-1} x)} .$$

Using the identity given in the hint, the expression $\tanh(\operatorname{sech}^{-1} x)$ becomes

$$\tanh(\operatorname{sech}^{-1} x) = \sqrt{1 - \operatorname{sech}^2(\operatorname{sech}^{-1} x)} = \sqrt{1 - x^2}.$$

We obtain

$$\frac{d}{dx} \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1 - x^2}} \quad , \quad 0 < x < 1 .$$

[15] 4. The field of *glottochronology* studies the development of languages over time. It is assumed that usage of a set of vocabulary words decreases at a rate proportional to the size N of the list: $N'(t) = kN(t)$, with k guessed at about -0.000217 and t measured in years. Suppose we choose a set of 210 common words in Latin and assume they were all in use when Italian broke off from Latin, and we find today that 144 of these words remain in Italian. Based on this data, about how long ago did Italian break off from Latin? (It is not necessary to find a decimal approximation for your answer.)

Solution. By assumption, N satisfies exponential decay; that is,

$$N(t) = N_0 e^{kt},$$

where N_0 represents the "point in time" that Italian and Latin split. N_0 is specified to be 210, and with the given value of k, this becomes

$$N(t) = 210e^{-0.000217t}.$$

We're looking for the current time, at which $N(t) = 144$, which gives the equation

$$144 = 210e^{-0.000217t}.$$

Thus $-0.000217t = \ln(144/210)$, which means

$$t = \frac{1}{0.000217}(\ln 210 - \ln 144) = \boxed{\frac{\ln(105/72)}{0.000217}}.$$

Note: To two significant digits (about as much as precision we could hope for with the given data), this gives $t \approx 1700$. That is, we've computed that Italian has been independent from Latin for about 1700 years. Of course, to get a more reliable picture, we'd need to run this experiment with several sets of words and apply statistics, and it would be good to look at other resources. The earliest texts in recognizable Italian dialects date from the

10th century.

[20] 5. One method of predicting behavior of long flexible molecules such as DNA is to use a freely jointed chain model (see Figure 4). Thermal forces lead to large shape fluctions in DNA preventing it from being fully extended. We would like to know how much force it takes to move the free end of the chain an average distance of X_{ave}. The equation for X_{ave} is

$$X_{ave} = nbL(F_k)$$

where n is the number of chain segments and F_k is proportional to the force F. The total length of the chain is nb and it is assumed to be 1.

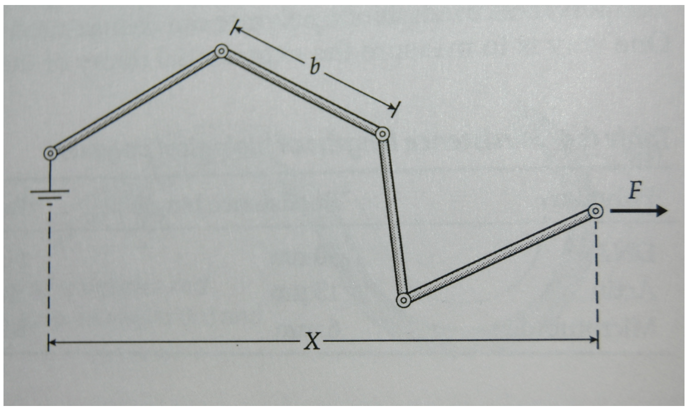


Figure 4: Schematic of the freely jointed chain model, courtesy J. Howard

(a) The function $L(F_k)$ is known to have the following form

$$L(F_k) = \frac{1}{Z}\int_0^{\pi} \cos\theta e^{F_k \cos\theta} \sin\theta d\theta$$

where $Z = \frac{1}{F_k}(e^{F_k} - e^{-F_k})$. Consider F_k to be a constant and evaluate the above integral.
Solution. Use the substitution $w = \cos\theta$, $dw = -\sin\theta d\theta$.

$$\int_0^{\pi} \cos\theta e^{F_k \cos\theta} \sin\theta d\theta = -\int_1^{-1} we^{F_k w} dw$$

Use integration by parts: $u = w$, $du = dw$ and $dv = e^{F_k w}dw$, $v = \frac{e^{F_k w}}{F_k}$. Also eliminate the negative sign by switching the limits.

$$\begin{aligned}
\int_{-1}^{1} we^{F_k w} dw &= \left[w\frac{e^{F_k w}}{F_k}\right]_{-1}^{1} - \int_{-1}^{1} \frac{e^{F_k w}}{F_k} dw \\
&= \left[\frac{e^{F_k}}{F_k} + \frac{e^{-F_k}}{F_k}\right] - \left[\frac{e^{F_k w}}{F_k^2}\right]_{-1}^{1} \\
&= \left[\frac{e^{F_k}}{F_k} + \frac{e^{-F_k}}{F_k}\right] - \left[\frac{e^{F_k}}{F_k^2} - \frac{e^{-F_k}}{F_k^2}\right] \\
&= \boxed{\frac{1}{F_k}(e^{F_k} + e^{-F_k}) - \frac{1}{F_k^2}(e^{F_k} - e^{-F_k})}
\end{aligned}$$

(b) Combine your solution from (a) with $\frac{1}{Z}$ to get an expression for $L(F_k)$ that involves a hyperbolic function.

Solution.

$$L(F_k) = \frac{\frac{1}{F_k}(e^{F_k} + e^{-F_k}) - \frac{1}{F_k^2}(e^{F_k} - e^{-F_k})}{\frac{1}{F_k}(e^{F_k} - e^{-F_k})} \tag{1}$$

Separate fractions to get

$$L(F_k) = \frac{(e^{F_k} + e^{-F_k})}{(e^{F_k} - e^{-F_k})} - \frac{1}{F_k} \tag{2}$$

We know $\cosh F_k = \dfrac{e^{F_k} + e^{-F_k}}{2}$ and $\sinh F_k = \dfrac{e^{F_k} - e^{-F_k}}{2}$ so

$$\lim_{F_k \to \infty} L(F_k) = \boxed{\coth F_k - \frac{1}{F_k}}$$

(c) Use your answer from part (b) to find the limit of X_{ave} as F_k goes to infinity.

Solution. Take the limit as F_k goes to infinity. Use the sum rule for limits in the first line.

$$\lim_{F_k \to \infty} L(F_k) = \lim_{F_k \to \infty} \frac{(e^{F_k} + e^{-F_k})}{(e^{F_k} - e^{-F_k})} - \lim_{F_k \to \infty} \frac{1}{F_k}$$

$$\lim_{F_k \to \infty} L(F_k) = \lim_{F_k \to \infty} \frac{(e^{F_k} + e^{-F_k})}{(e^{F_k} - e^{-F_k})} - 0$$

Now its easy to see that e^{-F_k} goes to zero as we take the limit. So it looks like the main contribution to the limit will be something like $\frac{e^{F_k}}{e^{F_k}}$, so we expect to get a limit of 1. Let's show this carefully. Mulitply by a form of 1 as shown below.

$$\begin{aligned}
\lim_{F_k \to \infty} L(F_k) &= \lim_{F_k \to \infty} \frac{(e^{F_k} + e^{-F_k})}{(e^{F_k} - e^{-F_k})} \cdot \left(\frac{e^{F_k}}{e^{F_k}}\right) \\
&= \lim_{F_k \to \infty} \frac{(e^{2F_k} + 1)}{(e^{2F_k} - 1)}
\end{aligned}$$

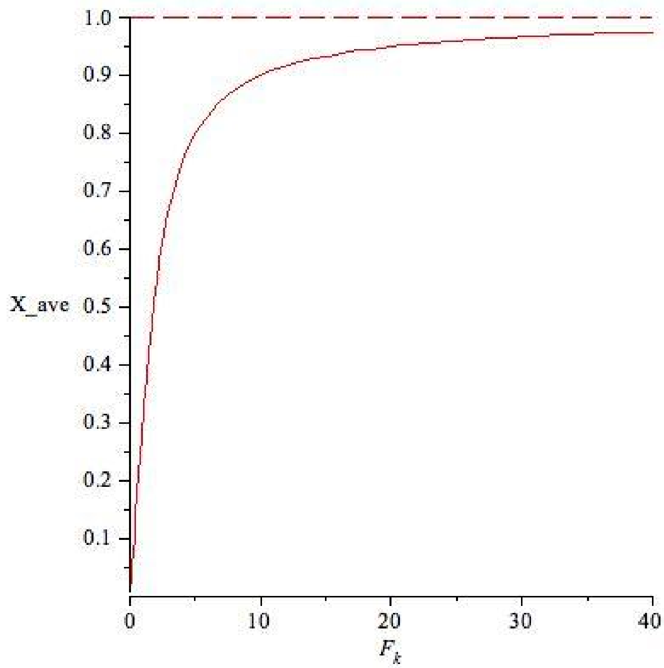
Taking the limit gives the indeterminant form $\frac{\infty}{\infty}$. Apply L'Hopital's rule once to obtain

$$\begin{aligned}
\lim_{F_k \to \infty} L(F_k) &= \lim_{F_k \to \infty} \frac{2e^{2F_k}}{2e^{2F_k}} \\
&= \lim_{F_k \to \infty} 1 \\
&= 1
\end{aligned}$$

Since $nb = 1$, we get that

$$\boxed{\lim_{F_k \to \infty} X_{ave} = 1}.$$

So we see the model predicts that it will take an infinite amount of force to fully extend the molecule. At lower forces this model does a reasonable job of predicting the behavior of DNA subjected to a tensile force. It is less accurate for large forces. The plot of X_{ave} vs F_k is given in the figure below.



1.0
0.9
0.8
0.7
0.6
X_ave
0.5
0.4
0.3
0.2
0.1
0
10
20
30
40
F_k