Your name: Solutions

Your TA's name: ______

Your section number and/or day and time: ______

Math 191, Prelim 3

Tuesday November 27, 2007. 7:30 – 9:00 PM

This exam should have 10 pages, with 5 problems adding up to 100 points.

The last 3 pages are blank and can be used as scrap paper for computations and checking answers.

No calculators or books allowed – You may have one 8.5×11 formula sheet.

To improve your chances of getting full credit (or maximum partial credit) and to ease the work of the graders, please:

- write clearly and legibly;
- $\boxed{\text{box in}}$ your answers;
- simplify your answers as much as possible;
- explain your answers as completely as time and space allow.

Problem 1:	/18
Problem 2:	/19
Problem 3:	/19
Problem 4:	/19
Problem 5:	/25
TOTAL:	/100

Academic Integrity is expected of all students of Cornell University at all times, whether in the presence or absence of members of the faculty.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

______

Signature of the Student

1. Find the following limits, **if they exist**. Justify your answers.

[4] a) $\lim_{n\to\infty} \frac{n+5}{\sqrt{2n^2-n+3}}$ $= \lim_{n\to\infty} \frac{n(1+5/n)}{n\sqrt{2-\frac{1}{n}+3/n^2}}$

$= \lim_{n\to\infty} \frac{1+5/n}{\sqrt{2-1/n+3/n^2}}$

$= \boxed{\frac{1}{\sqrt{2}}}$

[5] b) $\lim_{n\to\infty} \frac{1-(-1)^n}{2}\left(1-\frac{1}{n}\right)$

all even terms are zero,
all odd terms approach 1.

diverges by oscillation.

[4] c) $\lim_{n\to\infty} \frac{\sin(n^3)+\ln(n^4)}{n^2}$ $= \underbrace{\lim_{n\to\infty} \frac{\sin n^3}{n^2}}_{0;\ |\sin n^3|<1,\ n^2\to\infty} + \underbrace{\lim_{n\to\infty} \frac{\ln(n^4)}{n^2}}_{\text{L'Hôpital}}$

$-\frac{1}{n^2} \le \frac{\sin n^3}{n^2} \le \frac{1}{n^2}$

$\downarrow \quad \therefore \downarrow \quad \downarrow$

$0 \quad\quad 0 \quad\quad 0$

sandwich

L'Hôpital: $= \lim_{n\to\infty} \frac{\frac{1}{n^4}\cdot 4n^3}{2n}$

$= \boxed{0}$

[5] d) $\lim_{n\to\infty} \frac{\log_a(n)}{\log_a(2n)}$, where $a>1$.

$\overset{L/H}{=} \lim_{n\to\infty} \frac{\frac{1}{n\ln a}}{\frac{2}{2n\ln a}} = \lim_{n\to\infty} 1 = \boxed{1}$

[7] 2. a) Show that the improper integral $\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx$ converges.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2}du = \lim_{b\to\infty} -\frac{1}{u}\Big]_{\ln 2}^{\ln b} = \lim_{b\to\infty} -\frac{1}{\ln b} - \frac{-1}{\ln 2} = \boxed{\frac{1}{\ln 2}}$$

$u = \ln x \qquad u(2) = \ln 2$

$du = 1/x\, dx \qquad u(b) = \ln b$

[7] b) Show that the improper integral $\int_2^{\infty} \frac{1}{x(\ln x)^2(5 + 2e^x)}\, dx$ converges.

$x(\ln x)^2(5+2e^x) \geq x(\ln x)^2 \quad$ for $x \geq 2$

So $0 \leq \frac{1}{x(\ln x)^2(5+2e^x)} \leq \frac{1}{x(\ln x)^2} \quad$ for $x \geq 2$

Since $\int_2^{\infty} \frac{1}{x(\ln x)^2}dx$ converges, so does $\int_2^{\infty} \frac{1}{x(\ln x)^2(5+2e^x)}dx$ by direct comparison test for integrals.

[5] c) Does the series $\sum_{n=2}^{\infty} \underbrace{\frac{1}{n(\ln n)^2(5+2e^n)}}_{a_n}$ converge? Justify your answer.

yes, by the integral test.

let $f(x) = \frac{1}{x(\ln x)^2(5+2e^x)}$

1. $f(x) \geq 0$ for $x \geq 2$
2. $f(x)$ is continuous for $x \geq 2$
3. $f(x)$ is decreasing for $x \geq 2$

and $f(n) = a_n$ for all $n \geq 2$.

Since $\int_2^{\infty} f(x)dx$ converges by part b, so does $\sum_{n=2}^{\infty} a_n$.

[9] 3. a) Find the Taylor series of the function $f(x) = x^2$ at $x = a$. Show that the Taylor series is not an infinite series and that it is independent of the center a.

$$f(a) = a^2 \quad f'(a) = 2a \quad f''(a) = 2 \quad \underset{k \geq 3}{f^{(k)}(a) = 0}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = a^2 + 2a(x-a) + \frac{2(x-a)^2}{2!}$$

$$= a^2 + 2ax - 2a^2 + x^2 - 2ax + a^2$$

$$= x^2$$

[5] b) Show that the Taylor polynomial of order 1 at $x = 1$ is $2x - 1$.

$$P_1(x) = a^2 + 2a(x-a)\Big|_{a=1} = 1 + 2(x-1) = 2x - 1$$

@ $a = 1$

[5] c) Show that the Taylor polynomials of order 2 at $x = 1$ and order 3 at $x = 1$ are identical.

$$P_3(x) = P_2(x) + \frac{f^{(3)}(a)(x-a)^3}{3!}$$

$$\underset{k \geq 3}{f^{(k)}(a) = 0} \Rightarrow P_3(x) = P_2(x)$$

$f(0)=\frac{1}{1}=1, \quad f(\frac{1}{2})=f(-\frac{1}{2})=\frac{1}{1+\frac{1}{4}}=\frac{1}{(5/4)}=\frac{4}{5}$

4. For $f(x)=\dfrac{1}{1+x^2}$ you are given that $\int_{-1}^{1} f(x)\,dx=\dfrac{\pi}{2}$. $f(1)=f(-1)=\frac{1}{1+1}=\frac{1}{2}$

[11] a) Using Simpson's Rule with $n=4$ subdivisions, *i.e.* a step size of $\Delta x=\dfrac{1}{2}$, calculate an approximation to the integral $\displaystyle\int_{-1}^{1}\frac{1}{1+x^2}\,dx$.

$$S=\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+y_4\right)$$

$$=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}+4\left(\frac{4}{5}\right)+2(1)+4\left(\frac{4}{5}\right)+\frac{1}{2}\right)$$

$$=\frac{1}{6}\cdot\frac{47}{5}=\frac{47}{30}$$

[2] b) Using your answer to part (a) write down a ratio of integers which approximate the value of π.

$$\pi\approx 2\cdot\frac{47}{30}=\frac{47}{15}$$

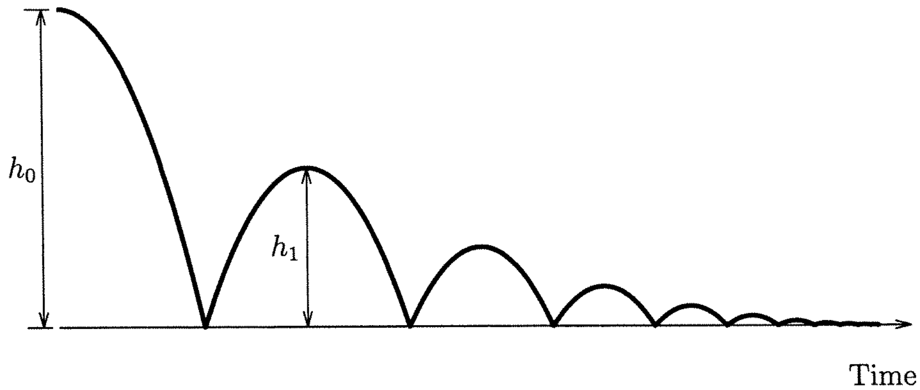
[6] c) Given that $\left|f^{(4)}(x)\right|\le 24$ for $-1\le x\le 1$ give an upper bound for the absolute difference between π and the fraction given in part (b).

$$\left|\frac{\pi}{2}-\frac{47}{30}\right|=|E_S|\le\frac{M(b-a)^5}{180n^4}=\frac{24\cdot 2^5}{180\cdot 4^4}=\frac{1}{60},$$

$$\text{so}\quad \left|\pi-\frac{47}{15}\right|=2\left|\frac{\pi}{2}-\frac{47}{30}\right|\le\frac{2}{60}=\frac{1}{30}$$

HINT: Remember the integral in part (a) is equal to $\dfrac{\pi}{2}$ **not** π.

5. Workshop Problem: **A bouncing ball**



Consider a ball that is dropped from a height h_0 on to a flat surface. Assume that at each rebound, the ball loses half of its mechanical energy. In this problem you will determine whether the ball ever stops bouncing or not. The initial mechanical energy of the ball is mgh_0, where m is the mass of the ball and g is the acceleration due to gravity. After the first rebound the ball rises back to a height h_1 such that

$$mgh_1 = \frac{1}{2}mgh_0, \quad i.e. \quad h_1 = \frac{1}{2}h_0.$$

Likewise in the second rebound the ball rises back to a height

$$h_2 = \frac{1}{2}h_1 = \left(\frac{1}{2}\right)^2 h_0.$$

[5] a) Is there a finite number of bounces $n < \infty$ after which the ball is at rest?

The height of the n^{th} bounce is $h_n = (\frac{1}{2})^n h_0 > 0$. Thus the ball has not stopped bouncing by the n^{th} bounce for any n. So there is not a finite number of bounces after which the ball is at rest.

20] b) The time taken for the ball to fall from height h_0 to the ground is given by $t_0 = \sqrt{2h_0/g}$, and the time elapsed between the n^{th} and $(n+1)^{\text{th}}$ rebounds is given by $t_n = 2\sqrt{2h_n/g}$, such that the total time elapsed before the $(n+1)^{\text{th}}$ bounce is

$$t_0 + t_1 + t_2 + \ldots + t_n = \sqrt{\frac{2h_0}{g}} + 2\sqrt{\frac{2h_1}{g}} + 2\sqrt{\frac{2h_2}{g}} + \ldots + 2\sqrt{\frac{2h_n}{g}}.$$

Does the ball ever come to rest?

The total time T elapsed while the ball bounces is

$$T = \sum_{n=0}^{\infty} t_n = \sqrt{\frac{2h_0}{g}} + \sum_{n=1}^{\infty} 2\sqrt{\frac{2\left(\frac{1}{2}\right)^n h_0}{g}}$$

$$= \sqrt{\frac{2h_0}{g}} + 2\sqrt{\frac{2h_0}{g}} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n,$$

which is finite because $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$ is a convergent geometric series.

Therefore the ball comes to rest (at time T).

HINT: Substitute $h_n = \left(\frac{1}{2}\right)^n h_0$ and identify a geometric series.