HOMEWORK 14 — SELECTED SOLUTIONS

Math 1920 — December 6, 2018

18.3

6) SOLUTION: div($\mathbf{F}$)=0, hence by the Divergence Theorem ($\mathcal{W}$ is the unit ball),

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{W}} \mathrm{div}(\mathbf{F}) \mathrm{d}V = 0$$

18) SOLUTION: First, let $\mathbf{F} = \langle x, 2y, 3z \rangle$. By the Divergence Theorem we can conclude:

$$\iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{W}} \mathrm{div}(\mathbf{F}) \mathrm{d}V = \iiint_{\mathcal{W}} 6 \mathrm{d}V = 6 \cdot \mathrm{Volume}(\mathcal{W})$$

Therefore, Volume($\mathcal{W}$)=12.

25) SOLUTION: The flow rate out of the sphere is

$$\iiint_{\mathcal{W}} \mathrm{div}(\mathbf{F}) \mathrm{d}V,$$

which may be approximated by

$$\mathrm{div}(\mathbf{F})(P) \cdot \mathrm{Vol}(\mathcal{W}) \approx 1.5708.$$

29) SOLUTION: Recall,

$$\mathrm{flux} = \iiint_{\mathcal{W}} \mathrm{div}(\mathbf{F}) \mathrm{d}V.$$

Using this fact we see:

$$\mathrm{flux} = \iiint_{\mathcal{W}} (-4) \mathrm{d}V = -4 \cdot \mathrm{Vol}(\mathcal{W}) = -4 \left(\frac{256\pi}{3} - 1 \right).$$

30) SOLUTION: div($\mathbf{F}$) $= 1$, then

$$\mathrm{flux} = \iiint_{\mathcal{W}} 1 \mathrm{d}V = \frac{76\pi}{3}.$$

35) SOLUTION:

a) We compute the divergence of $\nabla\varphi$:

$$\mathrm{div}(\nabla\varphi) = \mathrm{div}\left(\left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial y} \right\rangle \right) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \Delta\varphi.$$

b) In part a) we showed that $\Delta\varphi = \mathrm{div}(\nabla\varphi)$. Therefore $\Delta\varphi = 0$ if and only if $\mathrm{div}(\nabla\varphi) = 0$. That is, φ is harmonic if and only if $\nabla\varphi$ is divergence free.

c) We are given that $\mathbf{F} = \nabla\varphi$, where $\Delta\varphi = 0$. In part b) we showed that div($\mathbf{F}$)=0. We now show that curl($\mathbf{F}$)=0. We have:

$$\text{curl}(\mathbf{F}) = \text{curl}(\nabla\varphi) = \langle \varphi_{zy} - \varphi_{yz}, \varphi_{xz} - \varphi_{zx}, \varphi_{yx} - \varphi_{xy} \rangle = \mathbf{0}.$$

d) Integrate each component with respect to the appropriate variable and check that $\varphi = \frac{x^2 z}{2} - \frac{y^2 z}{2}$ is a potential function for the vector field $\mathbf{F} = \nabla\varphi$. It is straightforward to check that this φ is harmonic. Since $\mathbf{F}$ is the gradient of a harmonic function, we know by part c) that div($\mathbf{F}$)=0. Therefore, by the Divergence Theorem, the flux of $\mathbf{F}$ through a closed surface is zero:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{W}} \text{div}(\mathbf{F}) \mathrm{d}V = 0.$$