Math 1920 Homework 2 Selected Solutions

13.4

PQ2)

The length of $\mathbf{e} \times \mathbf{f}$ is

$$\|\mathbf{e}\| \, \|\mathbf{f}\| \sin\theta$$

where $\theta = {}^{\pi}/_{6}$ is the angle between them. As $\mathbf{e}$ and $\mathbf{f}$ are unit vectors this quantity is

$$1 \cdot 1 \cdot \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

PQ4)

In both cases the answer is $\mathbf{0}$, because, in both cases the angle between the vectors is 0, i.e., the vectors are parallel.

PQ6)

$\mathbf{v} \times \mathbf{w} = \mathbf{0}$ when either $\mathbf{v}$ or $\mathbf{w}$ are equal to $\mathbf{0}$ or $\mathbf{v}$ and $\mathbf{w}$ are parallel.

24)

We know that the length of $\mathbf{v} \times \mathbf{w}$ is $\|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta = 3 \cdot 3 \cdot {}^{1}/_{2}$. We also know that both of $\mathbf{v}$ and $\mathbf{w}$ are in the xz-plane and so a vector parallel to both of then is $\mathbf{k}$ (or anything parallel to it). Finally we know that $\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}$ is right handed. From the diagram this means that $\mathbf{v} \times \mathbf{w}$ points in the negative y direction:

$$\mathbf{v} \times \mathbf{w} = \frac{9}{2} \cdot (-\mathbf{k}) = -\frac{9}{2}\mathbf{k}$$

36)

From the figure $\mathbf{u} = \langle 1, 0, 4 \rangle$, $\mathbf{v} = \langle 1, 3, 1 \rangle$, and $\mathbf{w} = \langle -4, 2, 6 \rangle$, and the volume of the parallelepiped spanned by these vectors is given by the (absolute value of the) triple scalar product

$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix} 1 & 0 & 4 \\ 1 & 3 & 1 \\ -4 & 2 & 6 \end{vmatrix} \\
&= 1\cdot(3\cdot 6 - 1\cdot 2) - 0\cdot(1\cdot 6 - 1\cdot(-4)) + 4(1\cdot 2 - 3\cdot(-4)) \\
&= 1\cdot 16 - 0 + 4\cdot 14 \\
&= 72
\end{aligned}$$

44)

The triangle spanned by vectors $\mathbf{v}$ and $\mathbf{w}$ has area $\|\mathbf{v}\times\mathbf{w}\|/2$, so we want to find two vectors spanning the triangle. The points of the triangle are $P = (1,1,5), Q = (3,4,3)$, and $R = (1,5,7)$ and so vectors spanning it are $\overrightarrow{PQ} = \langle 2,3,-2\rangle$ and $\overrightarrow{PR} = \langle 0,4,2\rangle$. We compute

$$\begin{aligned}
\overrightarrow{PQ}\times\overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ 0 & 4 & 2 \end{vmatrix} \\
&= \mathbf{i}(3\cdot 2 - (-2)\cdot 4) - \mathbf{j}(2\cdot 2 - (-2)\cdot 0) + \mathbf{k}(2\cdot 4 - 3\cdot 0) \\
&= 14\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}
\end{aligned}$$

its length is $\sqrt{14^2 + (-4)^2 + 8^2} = 2\sqrt{69}$ and so the area of the triangle is $\sqrt{69}$.

13.5

PQ2)

$\mathbf{k}$ is normal to plane (c). A normal to (a) is $\mathbf{i}$ and to (b) is $\mathbf{j}$.

PQ4)

$y = 1$ is parallel to the plane $y = 0$ which is precisely the xz-plane.

28)

The plane must contain $P = (-1,0,1)$ and $\mathbf{r}(t) = \langle t+1, 2t, 3t-1\rangle$. A direction vector for $\mathbf{r}$ is $\langle 1,2,3\rangle$ (from the coefficients of t), and we know this is parallel to the plane. Also for every value of t, $\overrightarrow{P\mathbf{r}(t)}$ is parallel to the plane, in particular $\overrightarrow{P\mathbf{r}(0)} = \langle 2,0,-2\rangle$ is another vector parallel to the plane. As these two vectors are not parallel, their cross-product is normal to the plane

$$\begin{aligned}
\langle 1,2,3\rangle \times \langle 2,0,-2\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 0 & -2 \end{vmatrix} \\
&= \langle -4, 8, -4\rangle
\end{aligned}$$

For convenience we scale this to $\mathbf{n} = \langle -1, 2, -1 \rangle$ and so an equation for the plane is

$$\mathbf{n} \cdot \langle x, y, z \rangle = d$$

for a value of d we need to find. We plug in P to compute it

$$d = \langle -1, 2, -1 \rangle \cdot \langle -1, 0, 1 \rangle = 0$$

and so

$$\mathbf{n} \cdot \langle x, y, z \rangle = -x + 2y - z = 0$$

is the equation.

42)

The intersection of the plane $x - z = 6$ and line $\mathbf{r}(t) = \langle 1, 0, -1 \rangle + t \langle 4, 9, 2 \rangle$ can be found by substituting the expressions from the equation of the line in that of the plane:

$$(1 + 4t) - (-1 + 2t) = 1 + 4t + 1 - 2t = 2 + 2t = 6$$

and so $t = 2$ is the t value so that $\mathbf{r}(t)$ is in the plane, i.e., the intersection is $\mathbf{r}(2) = \langle 9, 18, 3 \rangle$.

54)

Say the plane has equation $ax + by + cz = d$. We know the intersection of this plane with the xy-plane is $\mathbf{r}(t) = t \langle 2, 1, 0 \rangle$, but we can also find this by setting $z = 0$ in the equation for the plane (as the equation for the xy-plane is $z = 0$) this tells us that

$$ax + by = d$$

must be the equation for the line parametrized by $\mathbf{r}$. The equation for this line can be recovered because we know $x = 2t, y = t, z = 0$ i.e. that $x = 2y$ and, as this line goes through the origin we know the d above is 0, and $b = -2a$. There is no constraint on c and so the planes are all of the form:

$$ax - 2ay + cz = 0.$$

66)

The intersection of $2x + y - 3z = 0$ and $x + y = 1$ can be found by substitution. Using the fact that $y = 1 - x$ from plane (2) we know

$$2x + (1 - x) - 3z = x - 3z + 1 = 0$$

Therefore $x = 3z - 1$ and so we can form the parametric equations:

$$z = t, \quad x = 3t - 1, \quad y = 1 - (3t - 1) = 2 - 3t$$