Math 1920 Homework 8 Selected Solutions

16.1

PQ2)

Assuming f is continuous on $\mathcal{R} = [0.9, 1.1] \times [1.9, 2.1]$ and $f(1,2) = 4$ we can approximate $\iint_{\mathcal{R}} f\,dA$ by assuming f is constant on $\mathcal{R}$, i.e.,

$$\iint_{\mathcal{R}} f\,dA \approx \int_{\mathcal{R}} 4\,dA = \text{area}(\mathcal{R}) \cdot 4 = (0.2) \cdot (0.2) \cdot 4 = 0.16$$

PQ4)

The interpretation of $\iint_{\mathcal{R}} f\,dA$ is the signed volume under the surface described by $z = f(x,y)$ over the region $\mathcal{R}$ in the xy-plane. Positive volume lies above this plane and negative lies below.

PQ6)

(b) and (c) are functions which integrate to 0 over the region by the inverse-symmetry of f over the y-axis, i.e., $f(x,y) = -f(-x,y)$. As the region is symmetrical in the y-axis the "positive part" of the integral cancels out the "negative part".

(a) and (d) both have positive integrals because the functions are almost always positive on the region.

18)

Firstly observe that $\iint_{\mathcal{R}} 2 + x^2y\,dA = \iint_{\mathcal{R}} 2\,dA + \iint_{\mathcal{R}} x^2y\,dA$. As $\mathcal{R} = [0,1] \times [-1,1]$, $\mathcal{R}$ has a line of symmetry in the x-axis, and, further, if we let $f(x,y) = x^2y$ we note that $f(x,y) = -f(x,-y)$, so f is "odd in the x-axis". Consequently, $\iint_{\mathcal{R}} x^2y\,dA = 0$ and so

$$\iint_{\mathcal{R}} 2 + x^2y\,dA = \iint_{\mathcal{R}} 2\,dA = \text{area}(\mathcal{R}) \cdot 2 = 4.$$

42)

Let $\mathcal{R} = [0,1] \times [1,2]$, then

$$\begin{aligned}
\iint_{\mathcal{R}} e^{3x+4y}\, dA &= \iint_{\mathcal{R}} e^{3x} \cdot e^{4y}\, dA \\
&= \left(\int_0^1 e^{3x}\, dx\right)\left(\int_1^2 e^{4y}\, dy\right) \\
&= \left[\frac{1}{3}e^{3x}\right]_0^1 \cdot \left[\frac{1}{4}e^{4y}\right]_1^2 \\
&= \frac{1}{12}(e^3 - 1)\cdot(e^8 - e^4) \\
&= \frac{1}{12}(e^{11} + e^4 - e^8 - e^7)
\end{aligned}$$

16.2

PQ2)

There are many, for instance, an annulus (a disk with a central disk cut out, like a flattened washer).

PQ4)

The maximum possible value is $4 \cdot \text{area}(\mathcal{D}) = 4\pi$, i.e., (b).

30)

$$\begin{aligned}
\int_0^1 \int_0^{\pi/2} x\cos(xy)\, dx\, dy = &\int_0^{\pi/2} \int_0^1 x\cos(xy)\, dy\, dx \\
&\int_0^{\pi/2} \sin(xy)\big|_{y=0}^1\, dx \\
&\int_0^{\pi/2} \sin x\, dx \\
&-\cos x\big|_0^{\pi/2} \\
&= -(0-1) \\
&= 1
\end{aligned}$$

42)

$f(x,y) = x+1$, and our region $\mathcal{R}$ is the triangle with vertices $(1,1), (5,3), (3,5)$. We split the region down the line $x = 3$. Our lower limit is the line going from $(1,1)$ to $(5,3)$, which has equation $2y = x+1$. Our upper limits have equations

$y = 2x - 1$ and $y = 8 - x$, the first for the left part of the region and the latter for the right. Using this we compute our integral

$$\begin{aligned}
\iint_{\mathcal{R}} f\,dA &= \iint_{\text{left region}} f\,dA + \iint_{\text{right region}} f\,dA \\
&= \int_1^3 \int_{\frac{x+1}{2}}^{2x-1} f\,dy\,dx + \int_3^5 \int_{\frac{x+1}{2}}^{8-x} f\,dy\,dx \\
&= \int_1^3 (2x-1)(x+1) - (\frac{x+1}{2})(x+1)\,dx + \int_3^5 (8-x)(x+1) - (\frac{x+1}{2})(x+1)\,dx \\
&= \int_1^3 2x^2 + x - 1 - x^2/2 - x - 1/2\,dx + \int_3^5 7x - x^2 + 8 - x^2/2 - x - 1/2\,dx \\
&= \int_1^3 \frac{3}{2}x^2 - \frac{3}{2}\,dx + \int_3^5 6x - \frac{3}{2}x^2 + \frac{15}{2}\,dx \\
&= 10 + 14 \\
&= 24
\end{aligned}$$

54)

The average y-coordinate is 0 by symmetry. For x we change to polar coordinates:

$$\begin{aligned}
\frac{2}{\pi R^2} \iint_{\text{semicircle}} x\,dA &= \frac{2}{\pi R^2} \int_{-\pi/2}^{\pi/2} \int_0^R r\cos\theta\, r\,dr\,d\theta \\
&= \frac{2}{\pi R^2} \int_{-\pi/2}^{\pi/2} \frac{r^3}{3} \cos\theta|_{r=0}^{R}\,d\theta \\
&= \frac{2}{\pi R^2} \int_{-\pi/2}^{\pi/2} \frac{R^3}{3} \cos\theta\,d\theta \\
&= \frac{2R}{3\pi} \sin\theta|_{-\pi/2}^{\pi/2} \\
&= \frac{2R}{3\pi} 2 \\
&= \frac{4R}{3\pi}
\end{aligned}$$

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16.3

PQ2)

(b) is not because the limits of the middle integral involve z, but we have already integrated with respect to z when you get around to computing the middle integral. (a) is fine.

10)

$$\iiint_{\mathcal{W}} f \partial V = \int_0^1 \int_0^1 \int_0^x e^{x+y+z} \, dy \, dz \, dx$$

16)

We find the equation of the upper surface: The vectors $\langle -4, 4, 0 \rangle$ and $\langle -4, 0, 6 \rangle$ live in the plane generated by the upper triangle, and so their cross product $8 \langle 3, 3, 2 \rangle$ is normal. An equation for the plane is then $3x + 3y + 2z = 12$ and so we can solve for $z = 6 - \frac{3}{2}x - \frac{3}{2}y$. Now we find the equation for the line in the xy-plane. This line has equation $3x + 3y = 12$ (plugging $z = 0$ into the eqn for the plane). We solve this for y and find $y = 4 - x$. This means we can express our integral as

$$\iiint_{\mathcal{V}} e^z \, dV = \int_0^4 \int_0^{4-x} \int_0^{6-\frac{3x}{2}-\frac{3y}{2}} e^z \, dz \, dy \, dx$$

which after a lengthy computation we find to be $\frac{1}{9}(4e^6 - 100)$.

26)

The upper surface is $z = 2 + x^2 + y^2$ and the lower surface is the plane $1 - x - y$. The projection of the region onto the xy-plane is the triangle enclosed by the line $y = 1 - x$ so our triple integral is

$$\int_0^1 \int_0^{1-x} \int_{1-x-y}^{2+x^2+y^2} f(x, y, z) \, dz \, dy \, dx$$

28)

(a) The upper face $z = 2 - y$ intersects the first quadrant of the xy-plane in the line $y = 2$ and so the projection of $\mathcal{W}$ onto the xy-plane is the triangle $\mathcal{D}$ defined by $0 \leq x \leq 1, 2z \leq y \leq 2$. Hence $\mathcal{W}$ is the region $0 \leq x \leq 1, 2x \leq y \leq 2, 0 \leq z \leq 2 - y$. So

$$\int_0^1 \int_{2x}^2 \int_0^{2-y} z \, dz \, dy \, dx$$

which can be show to be equal to $1/3$.

(b) For the yz-plane the projection is given by the region bounded by $y+z = 2$ and the positive y and z axes, i.e., $0 \leq y \leq 2$ and $0 \leq z \leq 2 - y$. We can bound x by $0 \leq x \leq y/2$ for

$$\int_0^2 \int_0^{2-y} \int_0^{y/2} z \, dx \, dz \, dy$$

which, of course, is also equal to $1/3$.

(c) Finally, we find the points on the intersection of the faces $2x - y = 0$ and $y + z = 2$. These are the points $(x, 2x, 2 - 2x)$ and so the projection of this onto the xz-plane is $(x, 0, 2 - 2x)$. This gives us inequalities $0 \leq z \leq 1$ and $0 \leq z \leq 2 - 2x$. For y we have $2x \leq y \leq 2 - z$ where $y = 2z$ is obtained bu the equation $y + z = 2$ of the upper face.

$$\int_0^1 \int_0^{2-2x} \int_{2x}^{2z} z \, dy \, dz \, dx$$

Which can be computed, tediously, to be $1/3$.