Problem 1. Consider the function

$$f(x,y) = \frac{4}{3}x^3 + y^2x - x.$$

(a) Find the critical points for this function

(b) Determine whether each of these is a local maximum, local minimum, or saddle point.

(c) Does $f(x,y)$ have a global maximum or global minimum? Why or why not?

Answer 1.1. (a) The critical point are points where the gradient of the function ∇f vanishes. Which then gives

$$\nabla f(x,y) = (4x^2 + y^2 - 1, 2xy) = 0.$$

Then we have two equations

$$4x^2 + y^2 = 1$$

and

$$xy = 0,$$

which result in four solutions

$$(x,y) = \left(\pm\frac{1}{2}, 0\right), \quad (0, \pm 1).$$

(b) Note that the discriminant is

$$D(x,y) = 16x^2 - 4y^2.$$

As a result, (1/2,0) is a local minima, (-1/2,0) is a local maxima, and $(0, \pm 1)$ is saddle points.

(c) It does not. Consider $x \to \infty$.

Problem 2. Use the method of Lagrange multipliers to find the point on the ellipsoid

$$\frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{4} = 1$$

that maximizes the sum $x + y + z$.

Answer 2.1. Let $f(x,y,z) = x + y + z$ and $g(x,y,z) = \frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{4}$. Then by the method of Lagrange multipliers, we need to find solutions of

$$\begin{aligned} 0 =& \nabla f(x,y,z) + \lambda \nabla g(x,y,z), \\ =& \left(1, 1, 1\right) + \lambda \left(\frac{2x}{3}, y, \frac{z}{2}\right). \end{aligned}$$

To find the values of λ, let us plug the solutions

$$(x,y,z) = -\frac{1}{\lambda}(\frac{3}{2}, 1, 2)$$

to

$$g(x,y,z) = 1.$$

Then we obtain

$$\frac{1}{\lambda^2} = \frac{4}{9}. \tag{1}$$

As a result, the maximum value of $f(x, y, z)$ is

$$f\left(1, \frac{2}{3}, \frac{4}{3}\right) = 3. \tag{2}$$

Problem 3. Consider the following iterated triple integral.

$$\int_0^2 \int_0^{y/2} \int_0^{2-y} xyz dz dx dy.$$

(a) Sketch the region of integration, and indicate the equations of the four bounding surfaces.

(b) Write this integral in the order of $dydzdx$. Do not evaluate this integral.

Answer 3.1. (a) Refers to Figure 1 and 2. Four bounding surfaces are

$$z = 0, \quad y = 0, \quad y + z = 2, \quad 2x - y = 0.$$

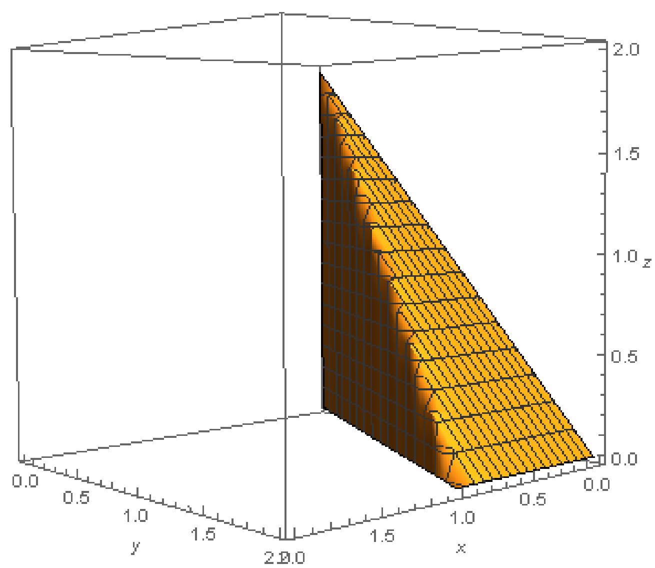


Figure 1:

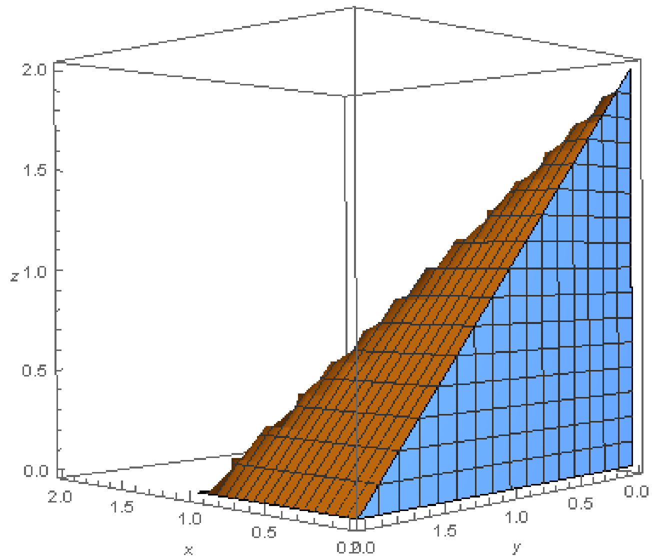


Figure 2:

(b) Given fixed x, z is bounded below $2-2x$ and bounded above 0. Similarly, y is bounded above $2-z$ and bounded below $2x$. Thus the domain of the integral is

$$\begin{aligned} 2x \leq y \leq 2-z, \\ 0 \leq z \leq 2-2x, \\ 0 \leq x \leq 1. \end{aligned}$$

As a result, the integral becomes

$$\int_0^1 \int_0^{2-2x} \int_{2x}^{2-z} xyz dydzdx.$$

Problem 4. Consider the hyperboloid given by

$$x^2 + 2y^2 - 3z^2 = 1.$$

Find all the points $P = (a, b, c)$ on the hyperboloid such that the tangen plane at P is parallel to the plane

$$3x + 2y + z = 0.$$

Answer 4.1. Let us denote $f(x, y, z) = x^2 + 2y^2 - 3z^2$. We need to find the points P such that

$$\begin{aligned} \lambda(3, 2, 1) =& \nabla f(P), \\ =& (2a, 4b, -6c). \end{aligned}$$

As a result, $(a, b, c) = (\frac{3\lambda}{2}, \frac{\lambda}{2}, -\frac{\lambda}{6})$. As the points P should be on the hyperboloid,

$$\left(\frac{3\lambda}{2}\right)^2 + 2\left(\frac{\lambda}{2}\right)^2 - 3\left(\frac{\lambda}{6}\right)^2 = 1.$$

Thus $\lambda = \pm\sqrt{6}/4$, which results in $(a, b, c) = \pm\sqrt{6}/4(3/2, 1/2, -1/6)$.

Problem 5. Let D be the region in space where $x > 0$, $y > 0$, and $z > 0$. Consider the vector field

$$F = \left\langle \frac{1}{x^2 y}, 1 + \frac{1}{xy^2}, \frac{1}{z} \right\rangle.$$

(a) Is D simply connected?

(b) Find curl F.

(c) Find a potential function for F, if one exists. If one doesn't exist, explain why.

Answer 5.1. (a) Yes. Every loop in D could be contractible to a point.

(b)

$$\text{curl} F = \begin{pmatrix} \text{i} & \text{j} & \text{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{x^2 y} & 1 + \frac{1}{xy^2} & \frac{1}{z} \end{pmatrix} = (0, 0, 0).$$

(c) As the curl vanishes and D is simply connected, the potential function exists.

$$V(x, y, z) - V(a, b, c) = -\int_{(a,b,c)}^{(x,y,z)} F \cdot dr,$$
$$= \frac{1}{xy} - \frac{1}{ab} - y + b - \ln\left(\frac{z}{c}\right).$$

Problem 6. Little Johnny has built a teleporter. Teleported objects materialize inside the half-ball

$$x^2 + y^2 + z^2 \leq 1, \quad z \geq 0.$$

The location of entry is random, with probability density $p(x, y, z) = kz^2$ (Note: $p(x, y, z) = 0$ outside the half-ball).

(a) What is the value of k which makes $p(x, y, z)$ a valid probabilty density function?

(b) What is the probability that the object materializes in the region above $z = \sqrt{x^2 + y^2}$ with $y \geq 0$?

Answer 6.1. (a) The probability density satisfies

$$\int\int\int p(x, y, z) dx dy dz = 1.$$

To find k, let us enumerate integral in the spherical coordinate.

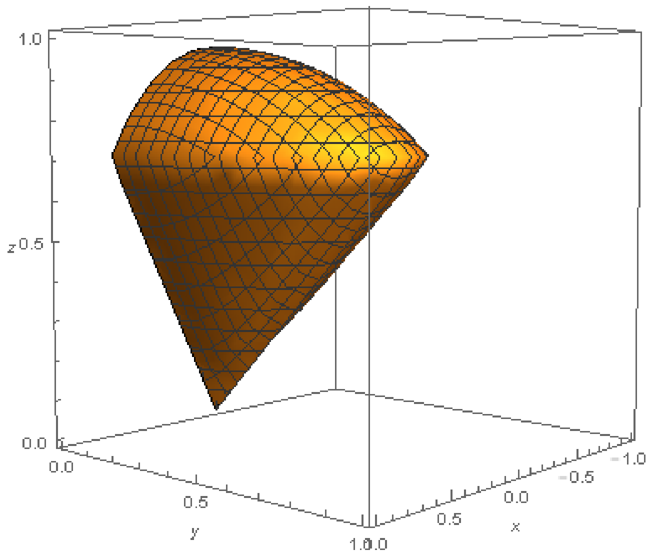
$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 k(\rho \cos\theta)^2 \rho^2 \sin\theta d\rho d\theta d\phi = \frac{2\pi}{15} k.$$

Thus $k = \frac{15}{2\pi}$.

(b) To simplify the computation, let us choose the cylindrical coordinate. Then the domain of integral is

$$r \le z \le \sqrt{1-r^2}, \quad 0 \le r \le \frac{\sqrt{2}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

As a result, the integral is



$$\int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{2}/2}\int_r^{\sqrt{1-r^2}} \frac{15}{2\pi} z^2 r dz dr d\theta = \frac{1}{8}(4-\sqrt{2}).$$