Math 1920	*Solutions*	
Final exam	12 December 2017	9:00–11:30 am

1 (10 points). *Suppose a function $g(u,v)$ satisfies $g(1,0)=2$ and $\nabla g(1,0)=\langle 2,-1\rangle$. Define a function $f(x,y,z)$ by $f(x,y,z)=g(e^{xz}+y,\sin(x-y+2z))$.*

(a) *Find $\nabla f(0,0,0)$.*

$f(x,y,z)=g(u,v)$, where $u=e^{xz}+y$, $v=\sin(x-y+2z)$. Use chain rule: $\frac{\partial f}{\partial x}=\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$ to get

$$\frac{\partial f}{\partial x}=ze^{xz}\frac{\partial g}{\partial u}+\cos(x-y+2z)\frac{\partial g}{\partial v}.$$

If $x=y=z=0$ then $u=1$ and $v=0$, so

$$\frac{\partial f}{\partial x}(0,0,0)=0+\cos(0)\frac{\partial g}{\partial v}(1,0)=(1)(-1)=-1.$$

Similarly,

$$\frac{\partial f}{\partial y}=\frac{\partial g}{\partial u}-\cos(x-y+2z)\frac{\partial g}{\partial v},\qquad \frac{\partial f}{\partial z}=xe^{xz}\frac{\partial g}{\partial u}+2\cos(x-y+2z)\frac{\partial g}{\partial v},$$

so

$$\frac{\partial f}{\partial y}(0,0,0)=2-\cos(0)(-1)=3,\qquad \frac{\partial f}{\partial z}(0,0,0)=0+2\cos(0)(-1)=-2.$$

Conclusion: $\nabla f(0,0,0)=\langle -1,3,-2\rangle$.

(b) *Find the equation of the tangent plane at $(0,0,0)$ to the surface $f(x,y,z)=2$.*

Normal vector is $\nabla f(0,0,0)=\langle -1,3,-2\rangle$ and plane goes through origin, so equation is $-x+3y-2z=0$.

2 (10 points). *Evaluate the double integral $\displaystyle\int_0^1\int_{\sqrt{x}}^1 xe^{y^5}\,dy\,dx$.*

Change order of integration: $\sqrt{x}\le y\le 1$, $0\le x\le 1$ becomes $0\le x\le y^2$, $0\le y\le 1$. So integral is

$$\int_0^1\int_0^{y^2} xe^{y^5}\,dx\,dy=\frac{1}{2}\int_0^1\Big[x^2e^{y^5}\Big]_{x=0}^{y^2}dy=\frac{1}{2}\int_0^1 y^4e^{y^5}\,dy=\frac{1}{2}\Big[\frac{1}{5}e^{y^5}\Big]_{y=0}^1=\frac{1}{10}(e-1).$$

3 (14 points). *Find the volume and the centroid of the solid region between the square $-1\le x\le 1$, $-1\le y\le 1$ in the xy-plane and the surface given by $z=xy+1$. (Recall that the centroid of a solid is the center of mass with respect to a constant mass density.)*

$|xy|\le 1$ for all points (x,y) in the square, so $xy+1\ge 0$. Hence the volume is

$$V=\int_{-1}^1\int_{-1}^1\int_0^{xy+1}dz\,dy\,dx=\int_{-1}^1\int_{-1}^1(xy+1)\,dy\,dx=\int_{-1}^1\Big[\frac{1}{2}xy^2+y\Big]_{y=-1}^1dx$$
$$=\int_{-1}^1\Big(\frac{1}{2}x+1-\Big(\frac{1}{2}x-1\Big)\Big)dx=\int_{-1}^1 2\,dx=4.$$

Centroid is $(\bar{x}, \bar{y}, \bar{z})$. Solid is symmetric under interchanging $x \leftrightarrow y$ and under interchanging $x \leftrightarrow -y$, so centroid lies on intersection of planes $x = y$ and $x = -y$. Hence $\bar{x} = \bar{y} = 0$. Also $\bar{z} = \frac{1}{V} \iiint z\, dV$, where

$$\begin{aligned}\iiint z\,dV &= \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{xy+1} z\,dz\,dy\,dx = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}(xy+1)^2\,dy\,dx \\ &= \int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{2}x^2y^2 + xy + \frac{1}{2}\right)dy\,dx = \int_{-1}^{1}\left[\frac{1}{6}x^2y^3 + \frac{1}{2}xy^2 + \frac{1}{2}y\right]_{y=-1}^{1} dx \\ &= \int_{-1}^{1}\left(\frac{1}{3}x^2 + 1\right)dx = \left[\frac{1}{9}x^3 + x\right]_{-1}^{1} dx = \frac{20}{9}.\end{aligned}$$

So centroid is $(0, 0, 5/9)$.

4 (10 points)**.** *Determine the local minima, local maxima and saddle points of the function* $f(x, y) = x^3 + y^2 - xy$.
Locate critical points by solving $\nabla f = \mathbf{0}$: $\nabla f = \langle 3x^2 - y, 2y - x\rangle$, so (x, y) is critical if and only if

$$3x^2 - y = 0, \qquad 2y - x = 0.$$

Subbing $y = 3x^2$ into second equation gives $6x^2 = x$, i.e. $x = 0$ or $x = 1/6$. So critical points are $P_1 = (0, 0)$ and $P_2 = (1/6, 1/12)$. Discriminant is

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix} = \begin{vmatrix} 6x & -1 \\ -1 & 2\end{vmatrix} = 12x - 1.$$

So $D(P_1) = -1 < 0$: P_1 is saddle point; whereas $D(P_2) = 1 > 0$ and $f_{xx}(P_2) = 1 > 0$: P_2 is local minimum.

5 (12 points)**.** *Let a, b and c be positive constants. Let $\mathscr{D}$ be the surface in $\mathbf{R}^3$ given by the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the inequalities $x \geq 0$, $y \geq 0$, $z \geq 0$. What is the maximum of the function $f(x, y, z) = xyz$ over the region $\mathscr{D}$? At what point of $\mathscr{D}$ is this maximum attained? (The answers will depend on a, b and c.)
$\mathscr{D}$ is closed and bounded and f is continuous, so f attains its global maximum somewhere on $\mathscr{D}$. The boundary $\partial\mathscr{D}$ of $\mathscr{D}$ is its intersection with each of the planes $x = 0$, $y = 0$, $z = 0$, so we have $f = 0$ on $\partial\mathscr{D}$. On the interior $\operatorname{int}\mathscr{D}$ of $\mathscr{D}$ the function f is positive, so the maximum must be attained on $\operatorname{int}\mathscr{D}$. To locate it, use Lagrange multiplier method: solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for $(x, y, z) \in \operatorname{int}\mathscr{D}$, where $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$. Since $\nabla f = \langle yz, xz, xy\rangle$ and $\nabla g = 2\langle x/a^2, y/b^2, z/c^2\rangle$ we get

$$a^2yz = 2\lambda x, \qquad b^2xz = 2\lambda y, \qquad c^2xy = 2\lambda z.$$

First equation gives $2\lambda = a^2yz/x$; substituting this into second and third equations gives $b^2xz = a^2y^2z/x$, $c^2xy = a^2yz^2/x$. Since $y \neq 0$, $z \neq 0$ this yields

$$x^2/a^2 = y^2/b^2 = z^2/c^2.$$

Therefore $g(x, y, z) = 3x^2/a^2$. The point (x, y, z) must be in $\mathscr{D}$, so $3x^2/a^2 = 1$, i.e. $x^2/a^2 = y^2/b^2 = z^2/c^2 = 1/3$. Conclusion: maximum is attained at $(a, b, c)/\sqrt{3}$; maximum value is $abc/\sqrt{27}$.

6 (12 points)**.** *True or false? If true, justify. If false, explain why or give a counterexample.*

(a) *If* $f(x, y)$ *is a continuous function, then* $\int_0^1 \int_0^x f(x, y)\, dy\, dx = \int_0^1 \int_0^y f(x, y)\, dx\, dy$.

False. Counterexample: let $f(x, y) = x - y$; then $\int_0^1 \int_0^x f(x, y)\, dy\, dx = \int_0^1 \frac{1}{2}x^2\, dx = \frac{1}{6}$, but $\int_0^1 \int_0^y f(x, y)\, dx\, dy = \int_0^1 -\frac{1}{2}y^2\, dy = -\frac{1}{6}$. (The reason for the discrepancy is that the two integrals refer to two different triangular regions in the plane.)

(b) *Let* $f(x, y)$ *be a function which is defined for all* x *and* y. *Let* $g(x, y) = (x^2 + y^2)^{2/3} f(x, y)$. *If* $f(x, y)$ *is continuous at the origin, then* $\frac{\partial g}{\partial x}(0, 0)$ *is well-defined and equal to* 0.

True.

Method 1. We have $g(0, 0) = (0^2 + 0^2)^{2/3} f(0, 0) = 0$, and hence the difference quotient is

$$\frac{g(h, 0) - g(0, 0)}{h} = \frac{h^{4/3} f(h, 0) - 0}{h} = h^{1/3} f(h, 0).$$

The partial derivative is by definition the limit of this difference quotient. This limit exists and is equal to

$$\frac{\partial g}{\partial x}(0, 0) = \lim_{h\to 0} \frac{g(h, 0) - g(0, 0)}{h} = \lim_{h\to 0}(h^{1/3} f(h, 0)) = \lim_{h\to 0} h^{1/3} \lim_{h\to 0} f(h, 0) = 0 f(0, 0) = 0.$$

Method 2. Assume that $\frac{\partial f}{\partial x}(x, y)$ exists and is continous for $(x, y) \neq (0, 0)$. (This assumption is not justified, so you will not receive full credit for this method.) Then by the product rule we have

$$\frac{\partial g}{\partial x}(x, y) = \frac{4x}{3(x^2 + y^2)^{1/3}} f(x, y) + (x^2 + y^2)^{2/3} \frac{\partial f}{\partial x}(x, y)$$

for $(x, y) \neq (0, 0)$. We have

$$\left| \frac{4x}{3(x^2 + y^2)^{1/3}} \right| \leq \frac{4|x|}{3|x|^{2/3}} = \frac{4}{3}|x|^{1/3},$$

so

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\partial g}{\partial x}(x, y) &= \lim_{(x,y)\to(0,0)} \frac{4x}{3(x^2 + y^2)^{1/3}} \lim_{(x,y)\to(0,0)} f(x, y) \\ &\quad + \lim_{(x,y)\to(0,0)} (x^2 + y^2)^{2/3} \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) \\ &= 0 f(0, 0) + 0 \frac{\partial f}{\partial x}(0, 0) = 0,\end{aligned}$$

which implies that $\frac{\partial g}{\partial x}(0, 0)$ exists and is equal to 0.

7 (12 points)**.** *True or false? If true, justify. If false, explain why or give a counterexample.*

(a) *Let* $f(x, y) = y - x$ *and let* $\mathscr{C}$ *be the unit circle* $x^2 + y^2 = 1$, *oriented in the counterclockwise direction. Then* $\int_{\mathscr{C}} \nabla f \cdot d\mathbf{r} = 2\pi$, *twice the area enclosed by* $\mathscr{C}$.

False. $\mathscr{C}$ is closed curve, so $\int_{\mathscr{C}} \nabla f \cdot d\mathbf{r} = 0 \neq 2\pi$ by Fundamental Theorem of Calculus in $\mathbf{R}^3$.

(b) *Suppose a vector field* $\mathbf{F}$ *defined on* $\mathbf{R}^3$ *is tangent to a closed surface* $\mathscr{S}$*, which is the boundary of a solid region* $\mathscr{W}$ *in* $\mathbf{R}^3$*. Then* $\iiint_{\mathscr{W}} \operatorname{div}(\mathbf{F})\, dV = 0$*.*

True. Divergence Theorem gives

$$\iiint_{\mathscr{W}} \operatorname{div}(\mathbf{F})\, dV = \iint_{\partial \mathscr{W}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F} \cdot \mathbf{n}\, dS.$$

Since $\mathbf{F}$ is tangent to $\mathscr{S}$ we have $\mathbf{F} \cdot \mathbf{n} = 0$, so $\iiint_{\mathscr{W}} \operatorname{div}(\mathbf{F})\, dV = \iint_{\mathscr{S}} \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

8 (14 points). (a) *Let* $\mathscr{C}$ *be the circle in the* xz*-plane given by* $x^2+z^2 = 1$, $y = 0$*, oriented counterclockwise when viewed from along the positive* y*-axis, and let*

$$\mathbf{F}(x, y, z) = \langle 3x^2 + 2xyz^2, 3y^2 + x^2z^2, 3z^2 + 2x^2yz \rangle.$$

Show that $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = 0$*.*

Method 1. Parametrize $\mathscr{C}$ by $\mathbf{r}(t) = \langle \cos t, 0, -\sin t \rangle$. Then $\mathbf{r}'(t) = -\langle \sin t, 0, \cos t \rangle$, so

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\langle 3\cos^2 t, \cos^2 t \sin^2 t, 3\sin^2 t \rangle \cdot \langle \sin t, 0, \cos t \rangle = 3\cos^2 t \sin t - 3\cos t \sin^2 t.$$

Hence $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (3\cos^2 t \sin t - 3\cos t \sin^2 t)\, dt = \left[-\cos^3 t - \sin^3 t\right]_0^{2\pi} = 0$.

Method 2. We have $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, whence $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathscr{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0$ by Stokes' Theorem, where $\mathscr{S}$ is any surface with boundary $\mathscr{C}$ (e.g. a circular disc in the xz-plane).

Method 3. We have $\mathbf{F} = \nabla f$, where $f(x, y, z) = x^3 + y^3 + z^3 + x^2yz^2$, so $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ by Fundamental Theorem of Calculus in $\mathbf{R}^3$.

(b) *Let*

$$\mathbf{F}(x, y, z) = \langle x\cos(xy) + x\sin(xz), xye^{xyz} - y\cos(xy), -z\sin(xz) - xze^{xyz} \rangle.$$

Show that $\iint_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S} = 0$ *for every closed oriented surface* $\mathscr{S}$ *in* $\mathbf{R}^3$*.*

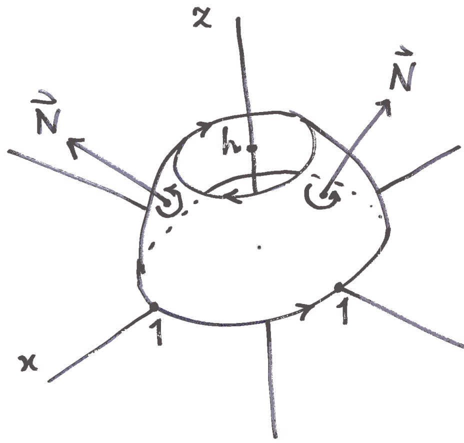
We have

$$\begin{aligned} \operatorname{div}(\mathbf{F}) &= \cos xy - xy\sin xy + \sin xz + xz\cos xz + xe^{xyz} + x^2yze^{xyz} \\ &\quad - \cos xy + yx\sin xy - \sin xz - zx\cos xz - xe^{xyz} - x^2zye^{xyz} \\ &= 0, \end{aligned}$$

so $\iint_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathscr{W}} \operatorname{div}(\mathbf{F})\, dV = 0$, where $\mathscr{W}$ is the bounded solid enclosed by $\mathscr{S}$.

9 (24 points). *Let* $\mathscr{S}$ *be the surface in* $\mathbf{R}^3$ *defined by* $x^2 + y^2 + z^2 = 1$ *and* $0 \le z \le h$*, where* h *is a constant satisfying* $0 \le h \le 1$*. Let* $\mathscr{S}$ *be oriented by the upward pointing normal vector.*

(a) *Sketch the surface* $\mathscr{S}$*. Include adequate detail such as coordinate axes and an indication of scale. In your sketch indicate the orientation of the boundary of* $\mathscr{S}$*.*



(b) *Parametrize the surface $\mathscr{S}$. Specify the domain $\mathscr{D}$ of the parametrization.*

Method 1. In cylindrical coordinates the sphere is given by $r^2 + z^2 = 1$, which leads to the parametrization $G(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$. The domain $\mathscr{D}$ of the parametrization is the annulus $0 \le \theta \le 2\pi$, $\sqrt{1-h^2} \le r \le 1$.

Method 2. In spherical coordinates the sphere is given by $\rho = 1$, which leads to the parametrization $G(\phi, \theta) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. The domain $\mathscr{D}$ of the parametrization is the rectangle $0 \le \theta \le 2\pi$, $\arccos h \le \phi \le \pi/2$.

(c) *Calculate the normal vector field* $\mathbf{N}$ *to the surface.*

Method 1.

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -r/\sqrt{1-r^2} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \cdots = \left\langle \frac{r^2}{\sqrt{1-r^2}}\cos\theta, \frac{r^2}{\sqrt{1-r^2}}\sin\theta, r \right\rangle.$$

Method 2.

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \end{vmatrix} = \cdots = \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \cos\phi\sin\phi \rangle.$$

(d) *Verify Stokes' Theorem for the surface $\mathscr{S}$ and the vector field*

$$\mathbf{F}(x, y, z) = \langle z - y, 0, y \rangle.$$

(The answers will depend on h.)

Part 1: the surface integral. We have $\operatorname{curl}(\mathbf{F}) = \langle 1, 1, 1 \rangle$.

Method 1.

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{N} = \frac{r^2}{\sqrt{1-r^2}}(\cos\theta + \sin\theta) + r$$

and

$$\iint_{\mathscr{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathscr{D}} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{N}\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_{\sqrt{1-h^2}}^1 \left(\frac{r^2}{\sqrt{1-r^2}} (\cos\theta + \sin\theta) + r \right) dr\, d\theta$$
$$= \int_0^{2\pi} \int_{\sqrt{1-h^2}}^1 r\, dr\, d\theta = 2\pi \left[\frac{1}{2} r^2 \right]_{r=\sqrt{1-h^2}}^1 = \pi h^2.$$

Method 2.

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{N} = \cos\theta \sin^2\phi + \sin\theta \sin^2\phi + \cos\phi \sin\phi$$

and

$$\iint_{\mathscr{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathscr{D}} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{N}\, dr\, d\theta$$
$$= \int_{\arccos h}^{\pi/2} \int_0^{2\pi} (\cos\theta \sin^2\phi + \sin\theta \sin^2\phi + \cos\phi \sin\phi)\, d\theta\, d\phi$$
$$= \int_{\arccos h}^{\pi/2} \int_0^{2\pi} \cos\phi \sin\phi\, d\theta\, d\phi = 2\pi \left[-\frac{1}{2} \cos^2\phi \right]_{\phi=\arccos h}^{\pi/2} = \pi h^2.$$

Part 2: the line integral. Two boundary pieces $\mathscr{C}_1$ and $\mathscr{C}_2$ parametrized by

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, 0 \rangle, \quad \text{resp.} \quad \mathbf{r}_2(t) = \langle \sqrt{1-h^2} \cos t, \sqrt{1-h^2} \sin t, h \rangle.$$

This parametrization of $\mathscr{C}_2$ goes the wrong way, so the integral over $\mathscr{C}_2$ must be taken with a negative sign. We have

$$\mathbf{r}_1'(t) = \langle -\sin t, \cos t, 0 \rangle, \qquad \mathbf{r}_2'(t) = \langle -\sqrt{1-h^2} \sin t, \sqrt{1-h^2} \cos t, 0 \rangle,$$

so

$$\mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t) = \langle -\sin t, 0, \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = \sin^2 t$$
$$\mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t) = \langle h - \sqrt{1-h^2} \sin t, 0, \sqrt{1-h^2} \sin t \rangle \cdot \langle -\sqrt{1-h^2} \sin t, \sqrt{1-h^2} \cos t, 0 \rangle$$
$$= -h\sqrt{1-h^2} \sin t + (1-h^2) \sin^2 t,$$

which yields

$$\int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \sin^2 t\, dt = \pi$$
$$\int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-h\sqrt{1-h^2} \sin t + (1-h^2) \sin^2 t)\, dt = \pi(1-h^2).$$

Conclusion: $\int_{\partial\mathscr{S}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{r} = \pi h^2 = \iint_{\mathscr{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

10 (22 points)**.** *Let* $\mathbf{A}$ *be the vector field in space defined by* $\mathbf{A} = \langle -yr^a, xr^a, 0 \rangle$*, where* a *is a constant and* $r = \sqrt{x^2 + y^2 + z^2}$*. Let* $\mathbf{F} = \operatorname{curl}(\mathbf{A})$*. (Note that for* $a < 0$ *the vector fields* $\mathbf{A}$ *and* $\mathbf{F}$ *are not defined at the origin.)*

(a) *Show that* $\mathbf{F} = (a+2)r^a\mathbf{k} - azr^{a-1}\mathbf{e}_r$, *where* $\mathbf{e}_r = \frac{1}{r}\langle x, y, z\rangle$ *is the unit radial vector field.*

Some useful formulas: $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = x/r$ (by quotient rule), so $\frac{\partial r^a}{\partial x} = ar^{a-1}\frac{\partial r}{\partial x} = axr^{a-2}$ (by chain rule). Also $\frac{\partial(xr^a)}{\partial x} = r^a + x\frac{\partial r^a}{\partial x} = r^a + ax^2r^{a-2}$ (by product rule). Similar rules hold for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. To summarize:

$$\frac{\partial r^a}{\partial x} = axr^{a-2}, \qquad \frac{\partial r^a}{\partial y} = ayr^{a-2}, \qquad \frac{\partial r^a}{\partial z} = azr^{a-2}$$

$$\frac{\partial(xr^a)}{\partial x} = r^a + ax^2r^{a-2}, \qquad \frac{\partial(yr^a)}{\partial y} = r^a + ay^2r^{a-2}, \qquad \frac{\partial(zr^a)}{\partial z} = r^a + az^2r^{a-2}.$$

It follows that

$$\begin{aligned}\mathbf{F} = \nabla\times\mathbf{A} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -yr^a & xr^a & 0\end{vmatrix} = \mathbf{i}\left(-x\frac{\partial r^a}{\partial z}\right) - \mathbf{j}\left(y\frac{\partial r^a}{\partial z}\right) + \mathbf{k}\left(\frac{\partial(xr^a)}{\partial x} + \frac{\partial(yr^a)}{\partial y}\right)\\ &= \langle -axzr^{a-2}, -ayzr^{a-2}, 2r^a + a(x^2+y^2)r^{a-2}\rangle\\ &= \langle -axzr^{a-2}, -ayzr^{a-2}, 2r^a + a(x^2+y^2+z^2)r^{a-2} - az^2r^{a-2}\rangle\\ &= \langle -axzr^{a-2}, -ayzr^{a-2}, (a+2)r^a - az^2r^{a-2}\rangle\\ &= (a+2)r^a\mathbf{k} - azr^{a-2}\langle x, y, z\rangle = (a+2)r^a\mathbf{k} - azr^{a-1}\mathbf{e}_r.\end{aligned}$$

(b) *Show that for* $a = -3$ *the vector field* $\mathbf{F}$ *is conservative.*

For $a = -3$ we have

$$\mathbf{F} = \langle 3xzr^{-5}, 3yzr^{-5}, -r^{-3} + 3z^2r^{-5}\rangle.$$

Method 1. Find a potential, i.e. a function f satisfying $\nabla f = \mathbf{F}$, i.e. solve the system of equations

$$\frac{\partial f}{\partial x} = \frac{3xz}{r^5}, \qquad \frac{\partial f}{\partial y} = \frac{3yz}{r^5}, \qquad \frac{\partial f}{\partial z} = -r^{-3} + 3z^2r^{-5}.$$

In part (a) we saw that $\frac{\partial r^a}{\partial x} = axr^{a-2}$. For $a = -3$ this means $\frac{\partial r^{-3}}{\partial x} = -3xr^{-5}$, so the general solution of the first equation is $f(x,y,z) = -zr^{-3} + h_1(y,z)$. Similarly, the general solution of the second equation is $f(x,y,z) = -zr^{-3} + h_2(x,z)$, We also have $\frac{\partial(zr^a)}{\partial z} = r^a + az^2r^{a-2}$, which for $a = -3$ gives $\frac{\partial(zr^{-3})}{\partial z} = r^{-3} - 3z^2r^{-5}$. So the general solution of the third equation is $f(x,y,z) = -zr^{-3} + h_3(x,z)$. Conclusion: $f(x,y,z) = -zr^{-3}$ is a potential of $\mathbf{F}$.

Method 2. Start by calculating the curl of $\mathbf{F}$:

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -axzr^{a-2} & -ayzr^{a-2} & (a+2)r^a - az^2r^{a-2}\end{vmatrix}.$$

The x-component of curl($\mathbf{F}$) is

$$\begin{aligned}(a+2)\frac{\partial r^a}{\partial y} - az^2\frac{\partial r^{a-2}}{\partial y} + ay\frac{\partial(zr^{a-2})}{\partial z} &= (a+2)ayr^{a-2} - az^2(a-2)yr^{a-4} + ay(r^{a-2} + (a-2)z^2r^{a-4})\\ &= (a+2)ayr^{a-2} - a(a-2)yz^2r^{a-4} + ayr^{a-2} + a(a-2)yz^2r^{a-4}\\ &= (a+3)ayr^{a-2},\end{aligned}$$

which is equal to 0 since $a = -3$. Similarly, the y-component of curl(**F**) is $(a+3)axr^{a-2} = 0$. The z-component of curl(**F**) is

$$\begin{aligned} -ayz\frac{\partial r^{a-2}}{\partial x} + axz\frac{\partial r^{a-2}}{\partial y} &= -ayz(a-2)xr^{a-4} + axz(a-2)yr^{a-4} \\ &= -a(a-2)xyzr^{a-4} + a(a-2)xyzr^{a-4} = 0. \end{aligned}$$

So curl(**F**) = **0**. The domain of **F** is punctured 3-space $\mathbf{R}^3 \setminus \{\mathbf{0}\}$, which is simply connected. Therefore **F** is conservative.