MATH 1920 - Fall 2018 - Prelim 1 Practice 1 Solutions

1. Since the particle has constant speed, $\|\mathbf{r}'(t)\|$ is constant. Thus $\|\mathbf{r}'(t)\|^2 = \mathbf{r}'(t)\cdot\mathbf{r}'(t)$ is also constant with respect to t. So,

$$\begin{aligned} 0 &= \frac{d}{dt}\mathbf{r}'(t)\cdot\mathbf{r}'(t) \\ &= \mathbf{r}''(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}''(t) && \text{(product rule)} \\ &= 2(\mathbf{r}'(t)\cdot\mathbf{r}''(t)). \end{aligned}$$

Then $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 0$, which means the velocity and acceleration vectors are always perpendicular to one another.

2. (a) The flux across the screen is $\overrightarrow{v}\cdot\overrightarrow{S}$ where $\overrightarrow{S} = S\overrightarrow{n}$, and S is the area of the screen and $\overrightarrow{n}$ is a unit vector perpendicular to the screen. We have

$$\begin{aligned} \overrightarrow{v} &= \langle 2,1,-1\rangle \\ S &= \pi(9)^2 = 81\pi \\ \overrightarrow{n} &= \frac{\langle 3,4,5\rangle}{\|\langle 3,4,5\rangle\|} = \frac{\langle 3,4,5\rangle}{\sqrt{3^2+4^2+5^2}} = \frac{1}{5\sqrt{2}}\langle 3,4,5\rangle. \end{aligned}$$

So the flux is

$$\begin{aligned} \overrightarrow{v}\cdot(S\overrightarrow{n}) &= \langle 2,1,-1\rangle\cdot\left((81\pi)\frac{1}{5\sqrt{2}}\langle 3,4,5\rangle\right) \\ &= \frac{81\pi}{5\sqrt{2}}(\langle 2,1,-1\rangle\cdot\langle 3,4,5\rangle) \\ &= \frac{81\pi}{5\sqrt{2}}((2)(3)+(1)(4)+(-1)(5)) \\ &= \frac{81\pi}{5\sqrt{2}}(5) \\ &= \frac{81\pi}{\sqrt{2}}. \end{aligned}$$

(b) Note that

$$\overrightarrow{v}\cdot(S\overrightarrow{n}) = S\|\overrightarrow{v}\|\|\overrightarrow{n}\|\cos\theta = S\|\overrightarrow{v}\|\cos\theta$$

where θ is the angle between $\overrightarrow{v}$ and $\overrightarrow{n}$. This is maximized when $\cos\theta = 1$, so $\theta = 0$. That is, the flow rate acoss the screen is as large as possible when the normal vector $\overrightarrow{n}$ to the screen is **parallel** to $\langle 2,1,-1\rangle$.

(c) $\mathbf{r}(t) = \langle 1,2,3\rangle + t\langle 2,1,-1\rangle$.

(d) We need to know if there exists t such that $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ lies in the plane

$3x + 4y + 5z = 6$. If this happens, then

$$\begin{aligned} 3x(t) + 4y(t) + 5z(t) &= 6 \\ 3(1+2t) + 4(2+t) + 5(3-t) &= 6 \\ 3 + 6t + 8 + 4t + 15 - 5t &= 6 \\ 26 + 5t &= 6 \\ 5t &= -20 \\ t &= -4. \end{aligned}$$

However, the bug only follows $\mathbf{r}(t)$ for $t \geq 0$, so we conclude that the bug **cannot** hit the screen if it continues down the river.

3. The three points form a syzygy when the vector from $\mathbf{r}_1(t)$ to $\mathbf{r}_2(t)$ is parallel to the vector from $\mathbf{r}_2(t)$ to $\mathbf{r}_3(t)$. Let

$$\begin{aligned} \mathbf{v} &= \overrightarrow{\mathbf{r}_1(t)\mathbf{r}_2(t)} = \langle 1, 3+t, -2-t \rangle \\ \mathbf{w} &= \overrightarrow{\mathbf{r}_2(t)\mathbf{r}_3(t)} = \langle t, -4-t, 0 \rangle \end{aligned}$$

These are parallel provided $\mathbf{w} = \lambda \mathbf{v}$. Then $\langle t, -4-t, 0 \rangle = \lambda \langle 1, 3+t, -2-t \rangle$. Component-wise,

$$\begin{cases} t &= \lambda \\ -4-t &= \lambda(3+t) \\ 0 &= \lambda(-2-t) \end{cases}$$

Plugging in $t = \lambda$ into the second equation, we get

$$\begin{aligned} -4 - t &= t(3+t) \\ 0 &= t^2 + 4t + 4 \\ 0 &= (t+2)^2 \quad \Longrightarrow \quad t = -2. \end{aligned}$$

We check that this satisfies the third equation, again substituting $\lambda = t$:

$$\lambda(-2-t) = t(-2-t) = (-2)(-2-(-2)) = (-2)(0) = 0. \quad \checkmark$$

So at time $t = -2$ the points form a syzygy.

4. The laser beam always shoots along the tangent vector to $\mathbf{r}(t)$. At time t_0, the tangent vector to $\mathbf{r}(t)$ is $\mathbf{r}'(t_0) = \langle -\sin t_0, \cos t_0 \rangle$. We can parametrize the path of the laser by

$$\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0) = \langle 2 + \cos t_0, \sin t_0 \rangle + t\langle -\sin t_0, \cos t_0 \rangle, \quad t \geq 0$$

We wish to find t such that $\mathbf{L}(t) = \langle 4, 0 \rangle$. That is, we want

$$\begin{cases} 2 + \cos t_0 - t \sin t_0 &= 4 \\ \sin t_0 + t \cos t_0 &= 0 \end{cases}$$

We solve the second equation for t to get $t = -\dfrac{\sin t_0}{\cos t_0}$. Plugging this into the first equation,

$$\begin{aligned} 2 + \cos t_0 - \left(-\frac{\sin t_0}{\cos t_0}\right)\sin t_0 &= 4 \\ 2\cos t_0 + \cos^2 t_0 + \sin^2 t_0 &= 4\cos t_0 \\ 2\cos t_0 + \cos^2 t_0 + (1 - \cos^2 t_0) &= 4\cos t_0 \\ 1 &= 2\cos t_0 \\ \cos t_0 &= \frac{1}{2}. \end{aligned}$$

Then either $t_0 = \frac{\pi}{3}$ or $t_0 = \frac{5\pi}{3}$. Also $t = -\dfrac{\sin t_0}{\cos t_0} = -\dfrac{\sin t_0}{1/2} = -2\sin t_0$. Since the laser beam only shoots forward, we must have $t \geq 0$. Then $\sin t_0 < 0$, which means only the solution $t_0 = \frac{5\pi}{3}$ works.

5. (a) Along $y = 0$:

$$\lim_{x\to 0} \frac{\sin(x \cdot 0)}{x^2 + 0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Along $y = x$,

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x \cdot x}{x^2 + ^2} &= \lim_{x\to 0} \frac{\sin(x^2)}{2x^2} \\ &= \lim_{x\to 0} \frac{\cos(x^2)\cdot 2x}{4x} \qquad \text{(L'Hôpital's Rule)} \\ &= \lim_{x\to 0} \frac{1}{2}\cos(x^2) \\ &= \frac{1}{2}\cos(0) \\ &= \frac{1}{2}. \end{aligned}$$

The limits are different along two different paths to $(0,0)$ so the limit does not exist.

(b) Method 1: In polar coordinates, we have

$$\begin{aligned} \lim_{r\to 0} \frac{(r\cos\theta)^2\sqrt{|r\sin\theta|}}{r^2} &= \lim_{r\to 0} \frac{r^2\cos^2\theta\sqrt{r}\sqrt{|\sin\theta|}}{r^2} \\ &= \lim_{r\to 0} \sqrt{r}\cos^2\theta\sqrt{|\sin\theta|} \\ &= 0 \end{aligned}$$

so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sqrt{|y|}}{x^2+y^2} = 0.$

Method 2: Note that $0 \leq \dfrac{x^2}{x^2+y^2} \leq 1$. Thus

$$0 \leq \frac{x^2\sqrt{|y|}}{x^2+y^2} \leq \sqrt{|y|}.$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} \sqrt{|y|} = \sqrt{|0|} = 0$, by the Squeeze Theorem we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sqrt{|y|}}{x^2+y^2} = 0.$$

6. (a)

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\ dt + \mathbf{c} \\ &= \int \langle t, \pi\cos(\pi t), 1\rangle\ dt + \mathbf{c} \\ &= \left\langle \frac{1}{2}t^2, \sin(\pi t), t\right\rangle + \langle c_1, c_2, c_3\rangle.\end{aligned}$$

Since $\mathbf{r}(0) = \langle 0, 1, -1\rangle$,

$$\begin{aligned}\langle 0,1,-1\rangle &= \left\langle \frac{1}{2}(0)^2, \sin(\pi\cdot 0), 0\right\rangle + \langle c_1, c_2, c_3\rangle \\ \langle 0,1,-1\rangle &= \langle 0,0,0\rangle + \langle c_1, c_2, c_3\rangle \\ \langle 0,1,-1\rangle &= \langle c_1, c_2, c_3\rangle.\end{aligned}$$

Thus $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, \sin(\pi t) + 1, t-1\right\rangle$.

(b) The speed $v(t)$ is the magnitude of the velocity vector:

$$v(t) = \|\mathbf{v}(t)\| = \|\langle t, \pi\cos(\pi t), 1\rangle\| = \sqrt{t^2 + \pi^2\cos^2(\pi t) + 1}.$$

(c) We need to know the values of t for which $\mathbf{r}(t) = \langle 0,1,-1\rangle$ and for which $\mathbf{r}(t) = \langle 8,1,3\rangle$. Since $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, \sin(\pi t), t-1\right\rangle$, we can look at the z-components to determine

$$\begin{aligned}\left\langle \frac{1}{2}t^2, \sin(\pi t), t-1\right\rangle &= \langle 0,1,-1\rangle \quad\Longrightarrow\quad t = 0 \\ \left\langle \frac{1}{2}t^2, \sin(\pi t), t-1\right\rangle &= \langle 8,1,3\rangle \quad\Longrightarrow\quad t = 4\end{aligned}$$

Then the distance traveled by the particle between these points is

$$\int_0^4 \|\mathbf{v}(t)\|\ dt = \int_0^4 \sqrt{t^2 + \pi^2\cos^2(\pi t) + 1}\ dt.$$

7. The intersection lies on the cylinder $y^2 + z^2 = 4$, so its projection onto the yz-plane lies on the circle $y^2 + z^2 = 4$. Thus we can take $y(t) = 2\cos(t)$ and $z(t) = 2\sin(t)$. Since we also want the curve to lie on the surface $x = y^2 z$, we must have $x(t) = y(t)^2 z(t) = 8\cos^2 t \sin t$. In order to trace the curve out exactly once, we choose $0 \le t < 2\pi$. Thus the parametrization is

$$\langle 8\cos^2 t \sin t, 2\cos t, 2\sin t \rangle, \qquad 0 \le t < 2\pi.$$

Note that there are many other parametrizations that are also correct.