MATH 1920 - Fall 2018 - Prelim 1 Practice 2 Solutions

1. (a) $\mathbf{v}$ is perpendicular to $\mathbf{w}$ if and only if their dot product is zero:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w}=0 \quad &\iff \quad (\mathbf{i}+2\mathbf{j}+a\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=0\\ &\iff \quad (1)(1)+(2)(1)+(a)(1)=0\\ &\iff \quad 3+a=0\\ &\iff \quad a=-3.\end{aligned}$$

(b) The area of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v}\times\mathbf{w}\|$.

$$\begin{aligned}\mathbf{v}\times\mathbf{w}&=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & a\\ 1 & 1 & 1\end{vmatrix}\\ &=(2\cdot 1-1\cdot a)\mathbf{i}-(1\cdot 1-1\cdot a)\mathbf{j}+(1\cdot 1-1\cdot 2)\mathbf{k}\\ &=(2-a)\mathbf{i}+(a-1)\mathbf{j}-\mathbf{k}\end{aligned}$$

So

$$\begin{aligned}\|\mathbf{v}\times\mathbf{w}\|=\sqrt{6} \quad &\iff \quad \|\mathbf{v}\times\mathbf{w}\|^2=6\\ &\iff \quad (\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})=6\\ &\iff \quad ((2-a)\mathbf{i}-(a-1)\mathbf{j}-\mathbf{k})\cdot((2-a)\mathbf{i}-(a-1)\mathbf{j}-\mathbf{k})=6\\ &\iff \quad (2-a)^2+(a-1)^2+(-1)^2=6\\ &\iff \quad 4-4a+a^2+a-3a+1+1=6\\ &\iff \quad 2a^2-6a=0\\ &\iff \quad 2a(a-3)0\\ &\iff \quad a=0 \text{ or } a=3.\end{aligned}$$

2. We are given that the plane is perpendicular to $\mathbf{v}=\langle 1,1,-4\rangle$, so $\mathbf{v}=\langle 1,1,-4\rangle$ is a normal vector to the plane. The plane also contains $(0,0,0)$, so an equation for the plane is

$$\begin{aligned}\langle 1,1,-4\rangle\cdot\langle x-0,y-0,z-0\rangle&=0\\ \text{i.e., } x+y-4x&=0\end{aligned}$$

We check that this plane is perpendicular to the plane $2x+2y+z=1$, i.e., their normal vectors $\langle 1,1,-4\rangle$ and $\langle 2,2,1\rangle$ are perpendicular:

$$\langle 1,1,-4\rangle\cdot\langle 2,2,1\rangle=(1)(2)+(1)(2)+(-4)(1)=0.\checkmark$$

3. (a) When $c=0$, the level curve of $g(x,y)$ is

$$\begin{aligned}\sqrt{y^2-x^2}&=0\\ y^2-x^2&=0\\ y^2&=x^2\\ y&=\pm x\end{aligned}$$

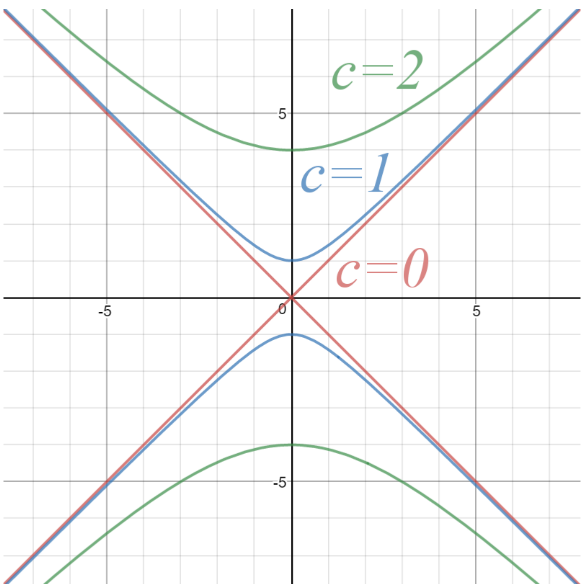
When $c = 1$, the level curve of $g(x, y)$ is

$$\begin{aligned} \sqrt{y^2 - x^2} &= 1 \\ y^2 - x^2 &= 1 \text{ (hyperbola opening along the } y\text{-axis)} \\ y^2 &= x^2 + 1 \\ y &= \pm\sqrt{x^2 + 1} \end{aligned}$$

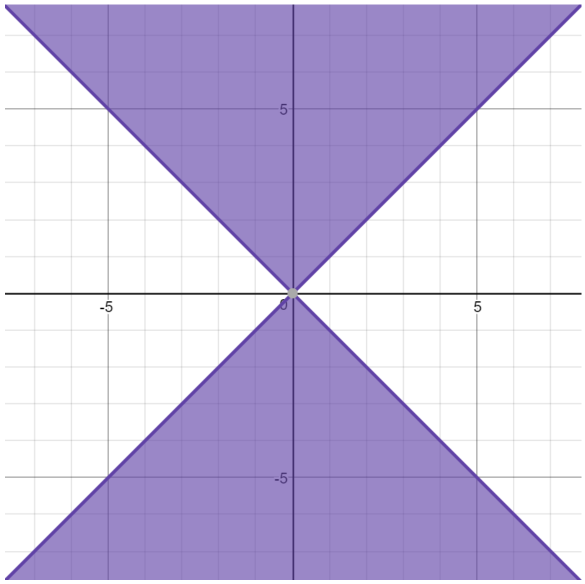
When $c = 2$, the level curve of $g(x, y)$ is

$$\begin{aligned} \sqrt{y^2 - x^2} &= 2 \\ y^2 - x^2 &= 4 \text{ (hyperbola opening along the } y\text{-axis)} \\ y^2 &= x^2 + 4 \\ y &= \pm\sqrt{x^2 + 4} \end{aligned}$$

These level curves are plotted below.



(b) We can't take the square root of a negative number, so we need $y^2 - x^2 \geq 0$. That is, $y^2 \geq x^2$, so $|y| \geq |x|$. This can also be written as the two inequalities $y \geq |x|$ or $y \leq -|x|$. The domain is plotted below.



4. We compute $\dfrac{\partial^2 u}{\partial t^2}$ and $\dfrac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial u}{\partial t} = \cos(x - at)(-a) = -a\cos(x - at)$$
$$\implies \frac{\partial^2 u}{\partial t^2} = -a \cdot -\sin(x - at)(-a) = -a^2 \sin(x - at)$$

$$\frac{\partial u}{\partial x} = \cos(x - at) \implies \frac{\partial^2 u}{\partial x^2} = -\sin(x - at)$$

So

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= -a^2 \sin(x - at) \\ &= a^2(-\sin(x - at)) \\ &= a^2 \frac{\partial^2 u}{\partial x^2}. \end{aligned}$$

Thus $u(x,t) = \sin(x - at)$ satisfies the wave equation.

5. (a) Note that

$$x - y - 1 = x - (y + 1) = (\sqrt{x} - \sqrt{y + 1})(\sqrt{x} + \sqrt{y + 1})$$

provided $x \geq 0$ and $y + 1 \geq 0$. If (x, y) is sufficiently close to $(4, 3)$, then indeed

$x \geq 0$ and $y+1 \geq 0$. So

$$\begin{aligned}
\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} &= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{(\sqrt{x}-\sqrt{y+1})(\sqrt{x}+\sqrt{y+1})} \\
&= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}} \\
&= \frac{1}{\sqrt{4}+\sqrt{3+1}} \\
&= \frac{1}{4}.
\end{aligned}$$

Note that we cannot approach along $x = y+1$ since the original function is not defined anywhere on that line.

(b) We approach along $x = 0$:

$$\begin{aligned}
\lim_{y\to 0} \frac{y}{\sqrt{0^2+y^2}} &= \lim_{y\to 0} \frac{y}{\sqrt{y^2}} \\
&= \lim_{y\to 0} \frac{y}{|y|}.
\end{aligned}$$

This limit does not exist since it depends on the sign of y as we approach 0 ($\lim_{y\to 0^+} \frac{y}{|y|} = 1$ but $\lim_{y\to 0^-} \frac{y}{|y|} = -1$). In order for $\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$ to exist, the limit must exist and be equal along all paths to $(0,0)$. Since we found a path along which the limit does not exist, $\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$ does not exist.

6. We find the partial derivatives of $g(x,y)$ at $(1,2)$:

$$g_x(x,y) = \frac{8x}{y} + \frac{2x}{x^2+y^2-4} \Longrightarrow g_x(1,2) = \frac{8\cdot 1}{2} + \frac{2\cdot 1}{1^2+2^2-4} = 6$$

$$g_y(x,y) = -\frac{4x^2}{y^2} + \frac{2y}{x^2+y^2-4} \Longrightarrow g_y(1,2) = -\frac{4\cdot 1^2}{2^2} + \frac{2\cdot 2}{1^2+2^2-4} = 3$$

We are given that $g(1,2) = 3$, but we can check:

$$g(1,2) = 1 + \frac{4\cdot 1^2}{2} + \ln(1^2+2^2-4) = 1+2+0 = 3.$$

So the tangent plane to the surface at $(1,2,3)$ is

$$\begin{aligned}
L(x,y) &= g(1,2) + g_x(1,2)(x-1) + g_y(1,2)(y-2) \\
&= 3 + 6(x-1) + 3(y-2) \\
&= 3 + 6x - 6 + 3y - 6 \\
&= 6x + 3y - 9.
\end{aligned}$$