MATH 1920 - Fall 2018 - Prelim 1 Practice 3 Solutions

1. (a) We find the partial derivaties of $f(x,y)$ at $(4,1)$:

$$f_x(x,y) = \frac{2x}{1+y^2} \implies f_x(4,1) = \frac{2\cdot 4}{1+1^2} = \frac{8}{2} = 4$$

$$f_y(x,y) = -\frac{x^2}{(1+y^2)^2}\cdot 2y \implies f_y(4,1) = -\frac{4^2}{(1+1^2)^2}\cdot 2\cdot 1 = -\frac{16}{2^2}\cdot 2 = 8$$

Next we find $f(4,1) = \frac{4^2}{1+1^2} = \frac{16}{2} = 8$. So the equation of the tangent plane has the form

$$\begin{aligned} L(x,y) &= f(4,1) + f_x(4,1)(x-4) + f_y(4,1)(y-1) \\ &= 8 + 4(x-4) - 8(y-1) \\ &= 4x - 8y. \end{aligned}$$

To put this plane in the form $ax+by+cz = d$, we write $z = 4x-8y$, so $4x-8y-z = 0$.

(b) The value of $f(x,y)$ at $(4.01, 0.98)$ is approximately equal to the value of $L(x,y)$ at $(4.01, 0.98)$ because $L(x,y)$ is the best linear approximation to $f(x,y)$ at $(4,1)$. So,

$$\begin{aligned} f(4.01, 0.98) &\approx L(4.01, 0.98) \\ &= 4(4.01) - 8(0.98) \\ &= 16.04 - 7.84 \\ &= 8.2. \end{aligned}$$

2. (a) Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two vectors in the plane containing A, B, and C, a normal vector to the plane is given by their cross product:

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\ &= \langle 0,-3,0\rangle \times \langle -1,2,1\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 0 \\ -1 & 2 & 1 \end{vmatrix} \\ &= [(-3)(1) - (2)(0)]\mathbf{i} - [(0)(1) - (-1)(0)]\mathbf{j} + [(0)(2) - (-1)(-3)]\mathbf{k} \\ &= \langle -3, 0, -3\rangle. \end{aligned}$$

The plane also contains the point $A = (1,1,1)$, so an equation for the plane is

$$\begin{aligned} \langle -3,0,-3\rangle \cdot \langle x-1, y-1, z-1\rangle &= 0 \\ -3(x-1) + 0(y-1) - 3(z-1) &= 0 \\ -3x + 3 - 3z + 3 &= 0 \\ -3x + 6 &= 3z \\ -x + 2 &= z. \end{aligned}$$

So $z = -x+2$ is the plane containing A, B, and C.

(b) If D is on the plane, then we must have $z = -x + 2$ for $x = 1$, $y = 2$, and $z = a$. That is, $a = -1 + 2$, so $a = 1$. This is the only value of a for which D lies on the plane containing A, B, and C.

(c) The area of the triangle formed by A, B, and C is half the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The area of that parallelograph is given by $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$. Thus area of the triangle formed by A, B, and C is

$$\begin{aligned} \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \frac{1}{2}\|\langle -3, 0, -3\rangle\| \\ &= \frac{1}{2}\sqrt{(-3)^2 + 0^2 + (-3)^2} \\ &= \frac{1}{2}\sqrt{18} \\ &= \frac{3}{2}\sqrt{2}. \end{aligned}$$

(d) To find the distance of D to the plane, we

- Compute the vector $\overrightarrow{AD}$
- Compute the projection $\text{proj}_{\mathbf{n}}(\overrightarrow{AD})$ of $\overrightarrow{AD}$ onto the normal vector $\mathbf{n}$ to the plane
- Compute the length of $\text{proj}_{\mathbf{n}}(\overrightarrow{AD})$

We have $\overrightarrow{AD} = \langle 0, 1, a - 1\rangle$, so

$$\begin{aligned} \text{proj}_{\mathbf{n}}(\overrightarrow{AD}) &= \left(\frac{\mathbf{n} \cdot \overrightarrow{AD}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} \\ &= \left(\frac{\langle -3, 0, -3\rangle \cdot \langle 0, 1, a - 1\rangle}{\langle -3, 0, -3\rangle \cdot \langle -3, 0, -3\rangle}\right)\langle -3, 0, -3\rangle \\ &= \left(\frac{-3(a-1)}{9+9}\right)\langle -3, 0, -3\rangle \\ &= \left(-\frac{a-1}{6}\right)\langle -3, 0, -3\rangle \\ &= \left\langle \frac{a-1}{2}, 0, \frac{a-1}{2}\right\rangle. \end{aligned}$$

Thus the distance from D to the plane is

$$\begin{aligned}
\|\text{proj}_{\mathbf{n}}(\overrightarrow{AD}) &= \left\|\left\langle \frac{a-1}{2}, 0, \frac{a-1}{2}\right\rangle\right\| \\
&= \sqrt{\left(\frac{a-1}{2}\right)^2 + 0^2 + \left(\frac{a-1}{2}\right)^2} \\
&= \sqrt{\frac{1}{4}(a-1)^2 + \frac{1}{4}(a-1)^2} \\
&= \sqrt{\frac{1}{2}(a-1)^2} \\
&= \frac{1}{\sqrt{2}}|a-1|.
\end{aligned}$$

We note that this distance is zero if and only if $a = 1$, i.e., D is on the plane if and only if $a = 1$ (as we found in part (b)).

3. When $\phi = 0$, $\rho = 4\cos(0) = 4$, and this corresponds to

$$\begin{cases} x = 4\sin(0)\cos\theta = 0 \\ y = 4\sin(0)\cos\theta = 0 \\ z = 4\cos(0) = 4 \end{cases}$$

So the point $(0, 0, 4)$ lies on the graph. When $\phi = \frac{\pi}{2}$, $\rho = 4\cos\left(\frac{\pi}{2}\right) = 0$, and this corresponds to

$$\begin{cases} x = 0\sin\left(\frac{\pi}{2}\right)\cos\theta = 0 \\ y = 0\sin\left(\frac{\pi}{2}\right)\sin\theta = 0 \\ z = 0\cos\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

So the point $(0, 0, 0)$ lies on the graph as well. Given that the equation describes a sphere centered on the z-axis, and given that the graph contains the points $(0, 0, 0)$ and $(0, 0, 4)$, it must be a sphere of radius 2 centered at $(0, 0, 2)$. We prove this by showing that every point on the graph satisfies $x^2 + y^2 + (z-2)^2 = 4$, the equation of a circe of radius 2 centered at $(0, 0, 2)$.

If the point (x, y, z) satisfies $\rho = 4\cos\phi$, then we have

$$\begin{aligned}
x &= \rho\sin\phi\cos\theta = (4\cos\phi)\sin\phi\cos\theta, \\
y &= \rho\sin\phi\sin\theta = (4\cos\phi)\sin\phi\sin\theta, \\
z &= \rho\cos\phi = (4\cos\phi)\cos\phi.
\end{aligned}$$

Thus

$$
\begin{aligned}
x^2 + y^2 + (z-2)^2 &= (4\cos\phi\sin\phi\cos\theta)^2 + (4\cos\phi\sin\phi\sin\theta)^2 + \left(4\cos^2\phi - 2\right)^2 \\
&= 16\cos^2\phi\sin^2\phi\cos^2\theta + 16\cos^2\phi\sin^2\phi\sin^2\theta + \left(4\cos^2\phi - 2\right)^2 \\
&= 16\cos^2\phi\sin^2\phi\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \left(4\cos^2\phi - 2\right)^2 \\
&= 16\cos^2\phi\sin^2\phi + (16\cos^4\phi - 16\cos^2\phi + 4) \\
&= 16\cos^2\phi\underbrace{(\sin^2\phi + \cos^2\phi)}_{1} - 16\cos^2\phi + 4 \\
&= 16\cos^2\phi - 16\cos^2\phi + 4 \\
&= 4.
\end{aligned}
$$

This shows that the graph of $\rho = 4\cos\phi$ lies entirely on the sphere of radius 2 centered at $(0,0,2)$. Moreover, as θ ranges from 0 to 2π and ϕ ranges from 0 to $\frac{\pi}{2}$, we see that the entire sphere will be traced out.

4. (a) We notice that $f(x,y)$ and $g(x,y)$ both have rotational symmetry around the z-axis: in polar coordinates we have $f(r,\theta) = \sqrt{3 - r^2}$ and $g(r,\theta) = \dfrac{r^2}{2}$, neither of which depend on θ. So the intersection of $f(x,y)$ and $g(x,y)$ should look like a circle in the xy-plane when we project it onto the xy-plane. Thus we can choose $x(t) = a\cos t$ and $y(t) = a\sin t$ for some constant a. Note then that $x(t)^2 + y(t)^2 = a^2$. We want $f(x(t),y(t)) = g(x(t),y(t))$, i.e.,

$$
\begin{aligned}
\sqrt{3 - x(t)^2 - y(t)^2} &= \frac{x(t)^2 + y(t)^2}{2} \\
\sqrt{3 - a^2} &= \frac{a^2}{2} \\
3 - a^2 &= \frac{a^4}{4} \\
0 &= a^4 + 4a^2 - 12 \\
0 &= (a^2 + 6)(a^2 - 2).
\end{aligned}
$$

Thus either $a^2 = -6$ or $a^2 = 2$. The first is impossible since $a^2 \geq 0$, so $a^2 = 2$ and we can choose $a = \sqrt{2}$. Finally, the z-coordinate of $\mathbf{r}_1(t)$ is

$$
f(x(t),y(t)) = g(x(t),y(t)) = \frac{(\sqrt{2}\cos t)^2 + (\sqrt{2}\sin t)^2}{2} = \frac{(\sqrt{2})^2}{2} = 1
$$

So $\mathbf{r}_1(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 1\rangle$.

(b) Note that $\langle 1,1,1\rangle = \mathbf{r}_1\left(\frac{\pi}{4}\right)$. We need a direction vector for the tangent line to $\mathbf{r}_1(t)$ at $t = \frac{\pi}{4}$, which is given by $\mathbf{r}_1'\left(\frac{\pi}{4}\right)$. We have

$$
\mathbf{r}_1'(t) = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0\rangle
$$

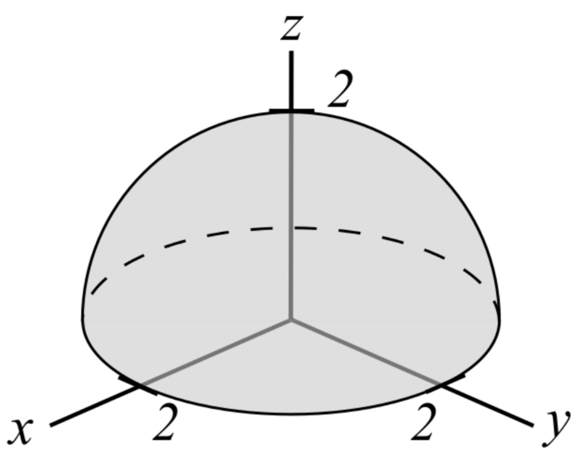
so

$$\mathbf{r}_1'\left(\frac{\pi}{4}\right) = \left\langle -\sqrt{2}\sin\left(\frac{\pi}{4}\right), \sqrt{2}\cos\left(\frac{\pi}{4}\right), 0\right\rangle = \langle -1, 1, 0\rangle$$

We also know the tangent line to $\mathbf{r}_1(t)$ at $(1,1,1)$ goes through the point $(1,1,1)$. So a parametrization of the tangent line is given by

$$\mathbf{r}_2(t) = \langle 1,1,1\rangle + t\langle -1,1,0\rangle$$

5. The spherical equations $\rho = 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \frac{\pi}{2}$ describe the upper hemisphere of a sphere of radius 2 centered at $(0,0,0)$, shown below.



6. In rectangular coordinates the equation for a sphere of radius 2 centered at the origin is $x^2 + y^2 + z^2 = 4$. We substitute the change of coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

to obtain

$$\begin{aligned} (r\cos\theta)^2 + (r\sin\theta)^2 + z^2 &= 4 \\ r^2 \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{1} + z^2 &= 4 \\ r^2 + z^2 &= 4. \end{aligned}$$

So in cylindrical coordinates the equation for a sphere of radius 2 centered at the origin is $r^2 + z^2 = 4$.

7. (a) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = y\sqrt{x^2+y^2} + xy\frac{2x}{2\sqrt{x^2+y^2}} = y\sqrt{x^2+y^2} + xy\frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = x\sqrt{x^2+y^2} + xy\frac{2y}{2\sqrt{x^2+y^2}} = x\sqrt{x^2+y^2} + xy\frac{y}{\sqrt{x^2+y^2}}$$

(b) Using the limit definition of partial derivatives,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h\cdot 0\sqrt{h^2+0^2} - 0\cdot 0\sqrt{0^2+0^2}}{h} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0\cdot h\sqrt{0^2+h^2} - 0\cdot 0\sqrt{0^2+0^2}}{h} = \lim_{h\to 0} 0 = 0$$

(c) In polar coordinates,

$$\lim_{r\to 0} \frac{r^2\cos^2\theta r\sin\theta}{r} = \lim_{r\to 0} r^2\cos^2\theta\sin\theta.$$

Note that $|\cos^2\theta\sin\theta| \leq 1$ for all θ, so $-1 \leq \cos^2\theta\sin\theta$. Thus

$$-r^2 \leq r^2\cos^2\theta\sin\theta \leq r^2.$$

We know that $\lim_{r\to 0} -r^2 = \lim_{r\to 0} r^2 = 0$, so by the squeeze theorem,

$$\lim_{r\to 0} r^2\cos^2\theta\sin\theta = 0.$$

So we conclude

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^2+y^2}} = 0.$$

(d) f_x and f_y are continuous at $(0,0)$ provided that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) = 0$$
$$\lim_{(x,y)\to(0,0)} f_y(x,y) = f_y(0,0) = 0$$

First we consider $f_x(x,y)$ in polar coordinates

$$\begin{aligned}\lim_{r\to 0} r\sin\theta\cdot r + r\cos\theta\cdot r\sin\theta\cdot\frac{r\cos\theta}{r} &= \lim_{r\to 0} r^2\sin\theta + r^2\cos^2\theta\sin\theta \\ &= \lim_{r\to 0} r^2(\sin\theta + \cos^2\theta\sin\theta)\end{aligned}$$

Since $|\sin\theta + \cos^2\theta\sin\theta| \leq 2$ for all θ, $-2 \leq \sin\theta + \cos^2\theta\sin\theta \leq 2$. Thus

$$-2r^2 \leq r^2(\sin\theta + \cos^2\theta\sin\theta) \leq 2r^2.$$

Since $\lim_{r\to 0} -2r^2 = \lim_{r\to 0} 2r^2 = 0$, by the squeeze theorem

$$\lim_{r\to 0} r^2(\sin\theta + \cos^2\theta\sin\theta) = 0.$$

That is, $\lim\limits_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$, so f_x is continuous at $(0,0)$.

Next we consider $f_y(x, y)$ in polar coordinates

$$\begin{aligned}\lim_{r\to 0} r\cos\theta \cdot r + r\cos\theta \cdot r\sin\theta \cdot \frac{r\sin\theta}{r} &= \lim_{r\to 0} r^2\cos\theta + r^2\cos\theta\sin^2\theta \\ &= \lim_{r\to 0} r^2(\cos\theta + \cos\theta\sin^2\theta)\end{aligned}$$

Again we have $|\cos\theta + \cos\theta\sin^2\theta| \leq 2$, so

$$-2r^2 \leq r^2(\cos\theta + \cos\theta\sin^2\theta) \leq 2r^2.$$

Again by the squeeze theorem

$$\lim_{r\to 0} r^2(\cos\theta + \cos\theta\sin^2\theta) = 0.$$

That is, $\lim_{(x,y)\to(0,0)} f_y(x, y) = f_y(0, 0)$, so f_y is continuous at $(0, 0)$.

Note that f_x and f_y (as found in part (a)) are continuous away from $(0, 0)$. We just showed that they are also continuous at $(0, 0)$. So f_x and f_y are continuous on an open disk containing $(0, 0)$, from which we can conclude that $f(x, y)$ is differentiable at $(0, 0)$.