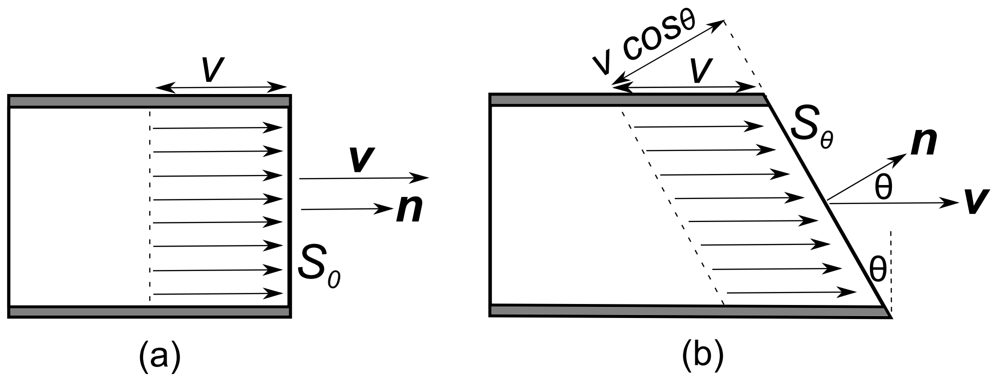
Problem 1 – Flux through a pipe

a) Assuming that the fluid velocity is the same over the entire cross-section, the volumetric flow rate is $vS_0 = 5$ m/s $\times$ 3 m^2 = 15 m^3/s.

Why? Consider a cylindrical column of fluid of area $S_0 = 3$ m^2 and length $v = 5$ m, as shown in part (a) of the figure below. This is the amount of liquid that will flow through S_0 in 1 second, as the fluid particles at the left edge will reach the end of the pipe (right edge) by that time. The volume of this liquid is $vS_0 = 15$ m^3, and hence the flow rate is $vS_0 = 15$ m^3/s.



b) Note that $S_0 = S_\theta \cos\theta$. This is easy to see if S_θ and S_0 were line segments as shown (S_0 is the vertical component/projection of S_θ). But it also works for areas, because you can imagine the area S_θ to be a collection of strips, with lengths in the plane of paper and infinitesimal thicknesses perpendicular to the plane of paper. S_0 is then the area composed of vertical projections of those strips. It is because of this that areas can be treated as vectors too.

The volumetric flow rate is the same as in **a)**. Here are two possible ways to see it:

1) **Intuitively**: Since the amount of fluid coming in from the left side of the pipe remains the same, the amount that comes out from the right side should remain unchanged regardless of the shape of the cut (Conservation of mass, with constant density).

2) **Mathematically**: Consider a slanted column of fluid of area S_θ and length v, as shown in the part (b) of figure above. This is the amount of liquid that will flow through S_θ in 1 second. The volume of this liquid is $vS_\theta \cos\theta$ (area time the height perpendicular to area). Hence the flow rate $vS_\theta \cos\theta = vS_0$.

Even though the area S_θ is larger than S_0 (by a factor of $\cos\theta$), the height/width of the slanted column is smaller than v by the same factor. Hence the volumes and flow rates, in part **a)** and **b)** are the same.

One can also imagine a deck of cards, with cards laid horizontally along the arrows shown in the figure. The deck in figure (b) is a displaced version of the deck in figure (a), and hence has the same volume (this is Cavalieri's principle).

c) In part a, the velocity field is $\vec{v} = v\hat{n}$ and the area vector is $\vec{S} = S_0\hat{n}$. The flux is then $\vec{v} \cdot \vec{S} = vS_0$.

In part b, the velocity field is the same: $\vec{v}$, but the area vector is $\vec{S} = S_\theta\hat{n}$, where $\hat{n}$ is at an angle θ with $\vec{v}$. The flux is then $\vec{v} \cdot \vec{S} = vS_\theta \cos\theta = vS_0$.

Flux $\vec{v}\cdot\vec{S}$ of a vector $\vec{v}$ over an area $\vec{S}$ is the magnitude of the area times the component of $\vec{v}$ along the normal direction. In part **b)** the area increases, but the component of the velocity normal to the area decreases by the same factor. The resulting flux is the same.

Problem 2 – A fun question to ponder

a) Our area vectors are given by $\|A_1\|\vec{j}$ and $\|A_2\|\vec{i}$ while $\vec{v} = -\|\vec{v}\|\vec{j}$ and $\vec{v} - \vec{w} = -\|\vec{v}\|\vec{j} - \|\vec{w}\|\vec{i}$.

b) The amount of water hitting A_1 and A_2 per unit time, i.e. the volumetric flow rates of the rain through your top and your front are then, respectively

$$Q_1 = \left(-\|\vec{v}\|\vec{j} - \|\vec{w}\|\vec{i}\right) \cdot \left(\|A_1\|\vec{j}\right) = -\|A_1\|\|\vec{v}\|$$

$$Q_2 = \left(-\|\vec{v}\|\vec{j} - \|\vec{w}\|\vec{i}\right) \cdot \left(\|A_2\|\vec{i}\right) = -\|A_2\|\|\vec{w}\|.$$

c) Assuming that you are moving in the direction of your house, the time it will take to travel distance s with velocity $\vec{w}$ is $t = \frac{s}{\|\vec{w}\|}$. The total amount of water hitting you as you travel to your home is given by

$$\begin{aligned} Q_1 t + Q_2 t &= -\|A_1\|\|\vec{v}\|\left(\frac{s}{\|\vec{w}\|}\right) - \|A_2\|\|\vec{w}\|\left(\frac{s}{\|\vec{w}\|}\right) \\ &= -\|A_1\|\|\vec{v}\|\left(\frac{s}{\|\vec{w}\|}\right) - \|A_2\|(s) \end{aligned}$$

d) Since the velocity $\vec{v}$ of the rain and the area A_1 are constant, the first term will be smaller when $\|\vec{w}\|$ is larger, so running will decrease the amount of water that will hit the top of your head. The second term in part *c*) does not depend on $\vec{w}$ at all, so you will hit the same amount of rain with your front no matter what speed you travel. In order to minimize how wet you get, it is better to run.

TAKE HOME POINTS – applications of the dot product

- Area vector $\vec{S}$ has magnitude equal to the area and direction normal to the area.
- Flux of a velocity field in a pipe is the same as the volumetric flow rate. It is independent of the cut made on the pipe, i.e independent of the direction of the area vector $\vec{S}$.
- Flux of a vector field $\vec{v}$ which is constant over an area $\vec{S}$ is $\vec{v} \cdot \vec{S}$.

- If the area is curved in 3D, and/or the vector field is not constant over the area, flux is a 'surface integral': $\int_S \vec{v} \cdot d\vec{S}$ of the vector field $\vec{v}$ through the surface S in space. It is the sum (integration) of the 'small fluxes' $(\vec{v} \cdot d\vec{S})$ through small area vectors $d\vec{S}$ on the surface S. We'll see surface integrals in the future.
- It is better to run in the rain!