Problem 1 – A simple example

a) We know that $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$. The magnitudes $\|\vec{u}\|$ and $\|\vec{v}\|$ are positive, so the sign of the dot product depends on the sign of $\cos\theta$. If the angle θ is acute the cosine will be positive and the dot product will be positive too. If the angle is obtuse then the cosine is negative and so is the dot product.

From A to D, the displacement vector $\vec{s}$ points straight down, while the force points upward and to the right. The angle between them is obtuse, so the dot product (work done) is negative. From D to C, the displacement and the force are perpendicular, so their dot product, and therefore the work done, is zero. Thus the work done along the path ADC is negative.

b) Applying the same process as part **a)**, the work done along the path ABC is positive, since from A to B the angle is acute and from B to C the vectors are perpendicular. Therefore the work done along ABC is not the same as the work done along ADC. Since we have one path from A to C where the work done is positive (ABC) and one where the work done is negative (ADC), we can tell that $\vec{F}$ is not a conservative force.

Problem 2 – Gravity

Since the paths CE and BD are perpendicular to the vector field, the dot product along these paths will be zero. Since the dot product is zero, the line integral and hence the work done by the force of gravity along these paths is 0. Therefore we need only consider the work done along AC, AB and DE.

Now the work done along AC can be thought of as the work done alone AB plus the work done along BC. Thus we need only compare the work done along BC to the work done along DE. Since our force field is radially symmetric, we can use this symmetry to determine that the work done along BC and DE is the same.

It turns out that $\vec{F}(x, y)$ is a conservative force, though we should note that the above argument does not prove this. In our problem we simply established that a certain two paths between a certain two points have the same work. Proving that the field is conservative in this fashion would require that we check all possible paths between all possible pairs of points. For obvious reasons, proving a field is conservative is usually done by other means that we may see later.

TAKE HOME POINTS – applications of the dot product

- Work done by a force $\vec{F}$, which is constant over a straight path of displacement $\vec{s}$ is $\vec{F} \cdot \vec{s}$.
- In general, work done is the 'line integral': $\int_A^B \vec{F} \cdot d\vec{s}$ of the (potentially varying) force $\vec{F}$ along a prescribed path from A to B. It is the sum (integration) of the 'small works' $(\vec{F} \cdot d\vec{s})$ done along the path. We'll see line integrals in the future.
- If the work done by a force field from point A to point B is independent of the path taken from A to B, then the force field is called conservative. Such force fields (e.g. gravity) are not uncommon and have special properties, to be seen in the future.