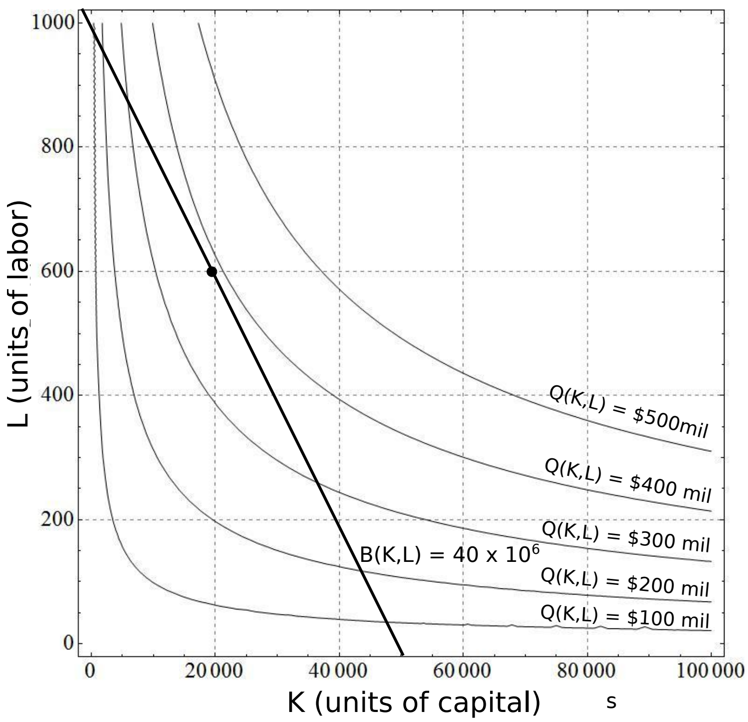
Math 1920, Workshop 4: Lagrange Multipliers in Economics: Solutions

(a) We will use Lagrange Multipliers to solve this problem i.e.

$$\nabla Q(K, L) = \lambda \nabla B(K, L)$$

Visually we are looking at the line $B(K, L) = 800K + 40,000L = 40 \times 10^6$ and trying to find where it is tangent to a level curve of $Q(K, L)$. The coordinates will give us our optimal K and L values, while the level curve we are tangent to will give us the maximum possible output.

(b) Drawing the line $B(K, L) = 40 \times 10^6$ on the graph below we can estimate that it will be tangent to a level curve at approximate coordinates $K = 20,000$ and $L = 6000$. The maximum profit i.e. the level curve which B is tangent to, is approximately \$390 million.



(c) Finding the gradients and using the lagrange conditions we get

$$\begin{aligned} (160,000)(.4)K^{-.6}L^{.6} &= 800\lambda & (160,000)(.6)K^{.4}L^{-.4} &= 40,000\lambda \\ 80(\tfrac{L}{K})^{.6} &= \lambda & 2.4(\tfrac{L}{K})^{-.4} &= \lambda \end{aligned}$$

Setting these equal we get

$$80\left(\frac{L}{K}\right)^{.6} = 2.4\left(\frac{L}{K}\right)^{-.4} \quad \text{i.e.} \quad \frac{K}{L} = \frac{100}{3} \quad \text{i.e.} \quad 3K = 100L$$

Plugging this into our equation for $B(K,L)$ we get

$$800K + 1200K = 40 \times 10^6 \quad \text{i.e.} \quad 2000K = 40 \times 10^6 \quad \text{i.e.} \quad K = 20,000$$

We can then solve for L to get $L = 600$ and so our maximum value for $Q(K,L)$ will occur at $(20,000, 600)$. Thus our visual estimate was spot on.

Extra Computation:
We can get a general solution for the critical points of

$$Q(K,L) = AK^{\alpha}L^{1-\alpha} \quad \text{and} \quad B(K,L) = pK + wL = M$$

First we must look at the gradients $\nabla Q = \lambda \nabla B$:

$$AaK^{a-1}L^{1-a} = \lambda p \quad \text{and} \quad A(1-a)K^{a}L^{-a} = \lambda w$$

Solving for p and w we get:

$$p\frac{Aa}{\lambda}K^{a-1}L^{1-a} \quad \text{and} \quad w = \frac{A(1-a)}{\lambda}K^{a}L^{-a}$$

Plugging these into our equation for $B(K,L) = M$ we get

$$\frac{Aa}{\lambda}K^{a}L^{1-a} + \frac{A(1-a)}{\lambda}K^{a}L^{1-a} = M \quad \text{or} \quad \frac{A}{\lambda}K^{a}L^{1-a} = M$$

Rearranging for λ we get

$$\lambda = \frac{A}{M}K^{a}L^{1-a}$$

Plugging this in to our original two equations and solving, we get the only critical point is

$$(K,L) = \left(\frac{Ma}{p}, \frac{M(1-a)}{w}\right)$$

TAKE HOME POINTS

- The local extrema of a differentiable function f can occur where $\nabla f = 0$ (or at the domain's boundary).
- If the variables also satisfy a constraint $g = 0$, then the extrema occur where the level curves of f are tangent to $g = 0$.
- If two curves are tangent, then their normals are parallel, i.e. scalar multiples of each other. Hence the extrema occur where $\nabla f = \lambda \nabla g$.
- λ is a scalar, called the Lagrange multiplier
- We need $g = 0$ along with $\nabla f = \lambda \nabla g$ to solve for the optimal variables and λ.
- NOTE: $\nabla f = \lambda \nabla g$ and $g = 0$, can be written as $\nabla(f - \lambda g) = 0$. Hence optimizing f with a constraint g is equivalent to optimizing $f - \lambda g$ with no constraint.