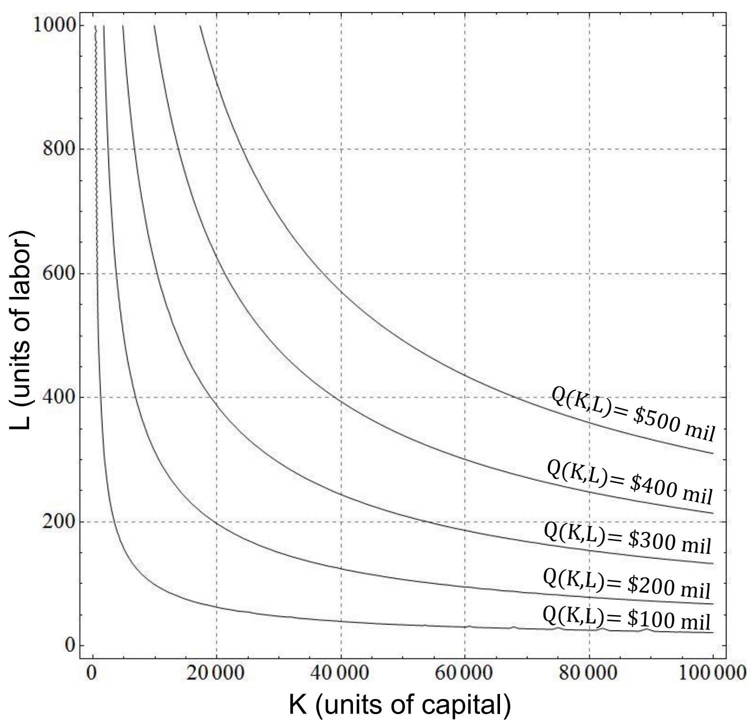
Math 1920, Workshop 4: Lagrange Multipliers in Economics

In 1920, C.W. Cobb and P.M. Douglas developed a model for the gross output, Q, of a nation or company given by the function

$$Q(K, L) = AK^{\alpha}L^{1-\alpha}$$

where K represents the amount of capital (consumable goods), L the about of Labor, and A and α are positive constants with $0 < \alpha < 1$. The figure shown below plots several *level curves* of $Q(K, L)$ where $A = 160,000$ and $\alpha = 0.4$ i.e.

$$Q(K, L) = 160,000K^{0.4}L^{0.6}$$



Remember that there are infinitely many level curves filling the graph for different values of Q, but we are only showing 5 here. These level curves are curves of constant output, and we can see that the output increases for level curves which are further from the origin. This makes sense because the more capital you start with and the more labor you are able to put in, the more money you are likely to make.

In order to think about maximizing profits, we need to think about budget constraints. Say each unit of labor costs w, each unit of capital costs p, and you only have a maximum amount M of money budgeted to invest in this venture. The budget constraint $B(K, L)$ can be represented as

$$B(K, L) = pK + wL = M.$$

(a) Suppose you have a company whose output is given by

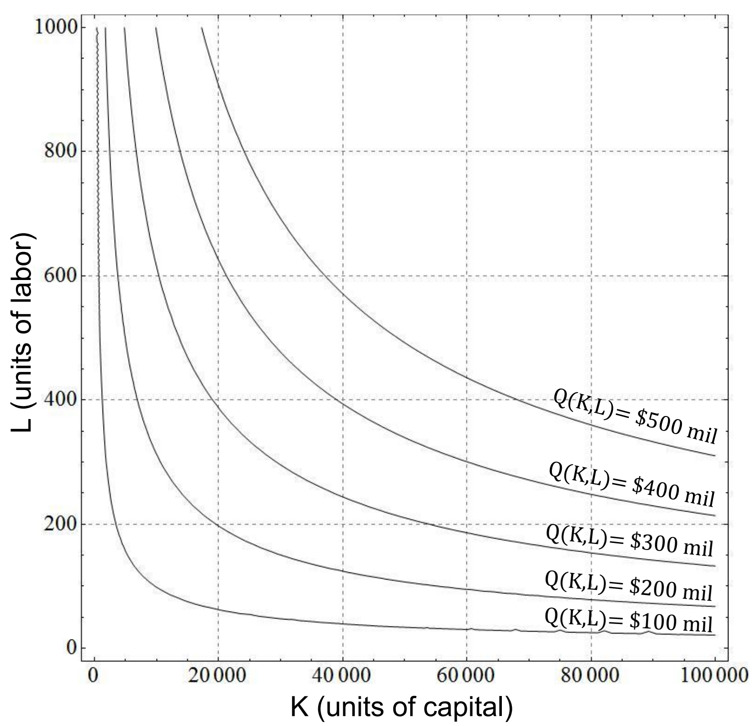
$$Q(K, L) = 160,000K^{0.4}L^{0.6}$$

(see the graph below) whose budget for the next fiscal year is given by

$$B(K, L) = 800K + (40,000)L = \$40 \times 10^6.$$

Our ultimate goal is to maximize profit, given the budget constraints. Write down the equation you will (eventually) use to solve this problem and then interpret it visually (i.e. how can we use the graph to visualize maximizing our function given a constraint). What point on the graph are we looking for?

(b) Use your answer to part (a) to get a visual estimate for the point on the graph which will maximize your profit (output), given the constraint. Write down the approximate coordinates and give an approximate value for the maximum output.



(c) Find the exact critical point using the equation you wrote down in part (a). How good was your visual estimate?

NOTE: It is possible to use Lagrange Multipliers to find the critical points for the general equations

$$Q(K, L) = AK^{\alpha}L^{1-\alpha} \qquad \text{and} \qquad B(K, L) = pK + wL = M$$

but the algebra is messy and tedious. A general solution will be posted in the workshop solutions, should you want to look at it.