Problem 1 – Probability

a) We want $\iint_D p(x,y)\,dx\,dy$ to be the probability of finding the particle in the rectangular region $R = [0,1] \times [1,0]$. Since the particle is always there, the value of the integral should be 1. Thus

$$\begin{aligned}
1 &= \int_0^1 \int_0^1 k \sin^2(2\pi x)\sin^2(2\pi y)\,dx\,dy \\
&= k\left(\int_0^1 \sin^2(2\pi x)\ dx\right)\left(\int_0^1 \sin^2(2\pi y)\ dy\right) \\
&= k\left[\frac{x}{2} - \frac{\sin(4\pi x)}{8\pi}\right]_0^1 \left[\frac{y}{2} - \frac{\sin(4\pi y)}{8\pi}\right]_0^1 \\
&= k\left[\frac{1}{2} - 0\right]\left[\frac{1}{2} - 0\right] \\
&= \frac{k}{4}
\end{aligned}$$

so that we must have $k = 4$.

b) To find the probability that the particle will be in a particular region we take the integral of the probability density function over that region:

$$\begin{aligned}
\int_0^{1/4}\int_0^{1/2} 4\sin^2(2\pi x)\sin^2(2\pi y)\,dx\,dy &= 4\left(\int_0^{1/2} \sin^2(2\pi x)\,dx\right)\left(\int_0^{1/4} \sin^2(2\pi y)\,dy\right) \\
&= 4\left[\frac{x}{2} - \frac{\sin(4\pi x)}{8\pi}\right]_0^{1/2} \left[\frac{y}{2} - \frac{\sin(4\pi y)}{8\pi}\right]_0^{1/4} \\
&= 4\left[\frac{1}{4} - 0\right]\left[\frac{1}{8} - 0\right] \\
&= \frac{1}{8}.
\end{aligned}$$

Thus the probability that the particle is in the region $[0, \frac{1}{2}] \times [0, \frac{1}{4}]$ is $\frac{1}{8}$.

c) The most probable locations are the ones with highest probability density. To see why, consider an infinitesimally small region of area dA around a point. The probability in that region is $p(x,y)\,dA$. Hence the probability is scaled by the value of $p(x,y)$ for infinitesimal regions around all points in R. Thus we need to maximize $p(x,y)$.

Option 1: Optimization techniques. Find the x and y where $\partial p/\partial x = 0$ and $\partial p/\partial y = 0$. Remember that this technique will give both maxima and minima, so you will need to classify the locations you determine through this method. Also remember to check the boundaries of R!

Option 2: We can save ourselves a lot of computation by using what we know about $\sin^2 t$. Note that $4\sin^2(2\pi x)\sin^2(2\pi y)$ is a product of 3 non-negative numbers, since $0 \le \sin^2 t \le 1$. Then this product will be maximized when $\sin^2(2\pi x)$ and $\sin^2(2\pi y)$ equal 1 (their maximum value) at the same location (note that depending on the functions involved, this may not always be possible – in which case you will have to resort to option 1).

$$\begin{aligned}
&\sin^2 t = 1 && t = \pi/2 + n\pi && = \pi/2, 3\pi/2, ... \\
&\sin^2(2\pi x) = 1 && x = \frac{\pi/2 + n\pi}{2\pi} && = \frac{1}{4}, \frac{3}{4} \\
&\sin^2(2\pi y) = 1 && y = \frac{\pi/2 + n\pi}{2\pi} && = \frac{1}{4}, \frac{3}{4}
\end{aligned}$$

All 4 combinations of x and y coordinates chosen from the options above, i.e. $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$, $(\frac{3}{4}, \frac{3}{4})$ will maximize the probability density with $p(x, y) = 4$. However the probability of finding the particle at any one of these points is zero because the region we would integrate over is a point.

Problem 2 – Expectation

Using the fact that we can separate the double integral into the multiplication of two integrals we find that

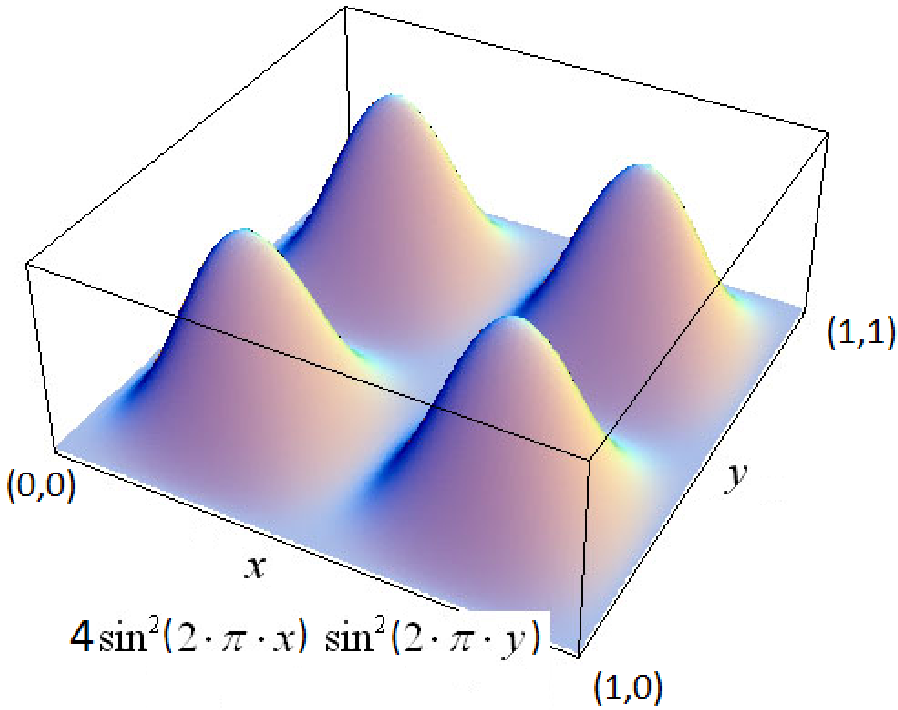
$$\begin{aligned}
E[x] &= \int_0^1 \int_0^1 4x \sin^2(2\pi x)\sin^2(2\pi y)\, dy\, dx \\
&= 4\left(\int_0^1 x\sin^2(2\pi x)\, dx\right)\left(\int_0^1 \sin^2(2\pi y)\, dy\right) \\
&= 4\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

and similarly

$$\begin{aligned}
E[y] &= \int_0^1 \int_0^1 4y \sin^2(2\pi x)\sin^2(2\pi y)\, dy\, dx \\
&= 4\left(\int_0^1 \sin^2(2\pi x)\, dx\right)\left(\int_0^1 y\sin^2(2\pi y)\, dy\right) \\
&= 4\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

So the expectation value of the particle's location is $(\frac{1}{2}, \frac{1}{2})$. Note that this is *not* one of the most probable locations. In fact, density here is 0.

In case you were curious, a graph of the probability density function is shown below. In the graph we can easily see the 4 peaks found in **1c)** and the symmetry that explains the expectation value from **2** being found in the center.



TAKE HOME POINTS – applications of probability density

- The integral of the probability density function $p(x,y)$ in a region D, $\iint_D p(x,y)\,dA$, is the probability of being in that region.
- If R is the entire domain then the probability of being anywhere in the domain R is 1, i.e. $\iint_R p(x,y)\,dA = 1$. Often a normalization constant (such as k in **1a)**) is used to ensure that a probability density function is normalized, which means exactly that the above condition holds.
- The expectation value of x is the weighted average $\iint_D x\,p(x,y)\,dA$
- Notice the parallel between probability density and mass density: expectation value is equivalent to the center of mass, and the most probable location is the position of maximum density. These are not necessarily the same.