# Math 1920 Workshop 5 : Probability Density

Subatomic particles exhibit wave like behavior. They are not 'located' at a particular location, but are 'spread-out' like waves in some sense. Quantum mechanics describes these particles using wave functions. If a measurement is performed in a region, the particles have some probability of being observed there, and this probability is provided by the squared magnitude of the wave function. We'll consider one simple model of a quantum mechanical particle which has a variety of applications.

Consider a particle confined to the two dimensional region $R = [0,1] \times [0,1]$ in the $(x, y)$ plane. For example, this could be a model of an electron on a patch of graphene. The *probability density* function is given below (see the note on probability density functions at the back). Integers $l$ and $n$ depend on the energy of the particle. Take $l = 2$ and $n = 2$ for this workshop.

$$p(x, y) = k \sin^2(l\pi x) \sin^2(n\pi y)$$

The 'volume' under the probability density function $p(x, y)$ over a subregion $D$ in $R$, i.e. $\iint_D p(x, y)\, dx\, dy$, is the probability of finding the particle in that particular subregion if a measurement is made.

**Useful integrals to save you time:**

$$\int \sin^2(m\pi z)\, dz = \frac{z}{2} - \frac{\sin(2m\pi z)}{4m\pi} + C_1$$

$$\int z \sin^2(m\pi z)\, dz = \frac{2m^2\pi^2 z^2 - \cos(2m\pi z) - 2m\pi z \sin(2m\pi z)}{8m^2\pi^2} + C_2$$

**Problem 1 – Probability**

**a)** What should the value of $\iint_R p(x, y)\, dx\, dy$ be in order for $p(x, y)$ to be a probability density function? Find the value of $k$ so that this condition holds.

**b)** When you make a measurement, what is the probability that the particle will be in the region $R_1 = [0, 1/2] \times [0, 1/4]$?

**c)** Find the most likely/probable location(s) for the particle to be. What is the probability density there? What is the probability that the particle will be at any one of these points when you make a measurement?

**Problem 2 – Expectation**
The *expectation value* of the position $(x, y)$, written as $(E[x], E[y])$ is the *average value* of $x$ and $y$ respectively, if the measurement was repeated infinitely many times. Expectation value can be calculated with the following formulas:

$$E[x] = \iint_R x\ p(x,y)\ dA \qquad \text{and} \qquad E[y] = \iint_R y\ p(x,y)\ dA$$

What is the expectation value of the particle's location? What is the probability density at that point?

**Note on Probability Density Functions:** For random experiments whose outcomes are finite in number, we can assign a probability to each outcome. For example, the experiment of throwing a dice has 6 outcomes, and the probability of each possible outcome is 1/6.

For experiments with infinite possible outcomes, the results can be expressed with continuous variables. For example, the particle in our 2D region can be found at any of the infinite locations given by $(x, y)$ values in $[0, 1] \times [0, 1]$. For these cases, we assign a probability density function to those continuous variables. The probability of finding the particle in the subregion $[a, b] \times [c, d]$ is the volume under the probability density function, given by $\int_a^b \int_c^d p(x, y)\, dy\, dx$.