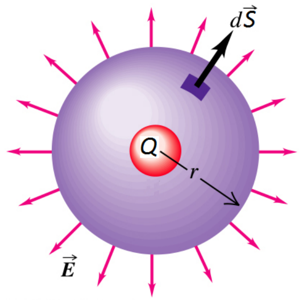
Problem 1 – Gauss's law

a) The figure below shows the electric field on the surface of the sphere.

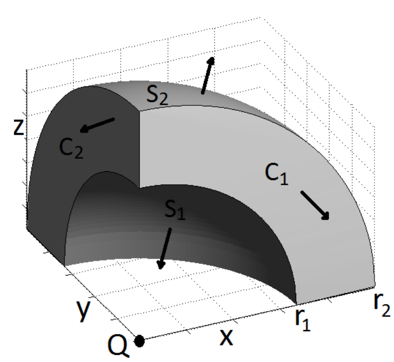


The magnitude of the electric field is constant over the surface of sphere with $||\vec{E}|| = \dfrac{Q}{4\pi\epsilon_0 r^2}$. Note that the area vectors (pointing normal to the sphere) are in the same direction as $\vec{E}$. Hence $\vec{E}\cdot d\vec{S} = ||\vec{E}||dS = \dfrac{Q}{4\pi\epsilon_0 r^2}dS$. Then

$$\oiint_S \vec{E}\cdot d\vec{S} = \iint_S \frac{Q}{4\pi\epsilon_0 r^2}dS = \frac{Q}{4\pi\epsilon_0 r^2}\iint_S dS = \frac{Q}{4\pi\epsilon_0 r^2}4\pi r^2 = \frac{Q}{\epsilon_0}.$$

b) If the charge were negative, the electric field would be in the opposite direction. Thus $\vec{E}\cdot d\vec{S} = -||\vec{E}||dS = -\dfrac{Q}{4\pi\epsilon_0 r^2}dS$ and the resulting flux will be $-\dfrac{Q}{\epsilon_0}$.

c) The surface S shown below can be separated into 5 parts: C_1, C_2 and C_3 in the coordinate planes xz, yz, and xy and S_1 and S_2 as parts of spheres with radii r_1 and r_2.



The flux is zero through the parts C_1, C_2 and C_3 because the radial electric field is perpendicular to the surface normals (hence $\vec{E}\cdot d\vec{S} = 0$). The flux through S_1 and S_2 are similar to **a)**, with the adjustment that we now have $\frac{1}{8}$ of the surface of a sphere. Thus

$$\iint_{S_1} \vec{E}\cdot d\vec{S} = \iint_{S_1} \frac{-Q}{4\pi\epsilon_0 r_1^2}dS = \frac{-Q}{4\pi\epsilon_0 r_1^2}\iint_{S_1} dS = \frac{-Q}{4\pi\epsilon_0 r_1^2}\left(\frac{1}{8}4\pi r_1^2\right) = \frac{-Q}{8\epsilon_0}$$

and

$$\iint_{S_2} \vec{E} \cdot d\vec{S} = \iint_{S_2} \frac{Q}{4\pi\epsilon_0 r_2^2} dS = \frac{Q}{4\pi\epsilon_0 r_2^2} \iint_{S_2} dS = \frac{Q}{4\pi\epsilon_0 r_2^2}\left(\frac{1}{8} 4\pi r_2^2\right) = \frac{Q}{8\epsilon_0}.$$

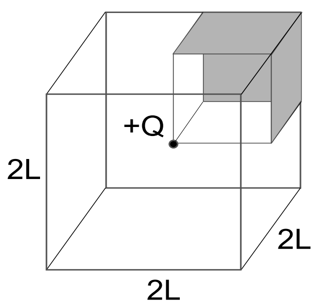
The total flux through all parts of S is thus 0.

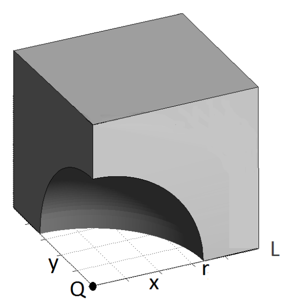
d) Gauss's law states that the flux of electric field through a closed surface is the net charge enclosed in the surface divided by ϵ_0:

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enclosed}}{\epsilon_0}$$

Imagine charges as the perpetual sources of flux, with $\frac{Q}{\epsilon_0}$ the amount of electric flux 'flowing' out of positive charges and 'flowing' into negative charges. For surfaces enclosing no net charge, whatever flux enters it, leaves it. No net flux is generated inside it. Even though we 'derived' Gauss's law for point charges and by integrating over simple surfaces, it is actually valid for any closed surface with any charge distribution.

e) *METHOD 1:* We realize that the surface asked for will not work, at least not in its original form. Gauss's law requires a *closed* surface, and the shaded sides don't form a closed surface. Instead, we choose to consider the surface of a cube of length $2L$, which has the charge Q located at its center (see the left figure below). Since the electric field is defined to be radially symmetric we are looking to create radial symmetry about the point charge.





The bigger cube is a closed surface enclosing the charge Q, hence Gauss's law applies. The integral that we are really interested in is, due to symmetry, just 1/8 of the flux through the entire closed surface. In short,

$$\oiint_{\text{big cube}} \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

so that

$$\iint_{\text{shaded}} \vec{E} \cdot d\vec{S} = \frac{1}{8} \oiint_{\text{big cube}} \vec{E} \cdot d\vec{S} = \frac{Q}{8\epsilon_0}.$$

METHOD 2: Consider the surface shown on the right above. The total flux is 0, because there is no charge inside. The flux through the spherical part is $-\frac{Q}{8\epsilon_0}$ as seen before in **c)**. The flux through the faces parallel to xy, yz, and xz planes is 0. Hence the flux through the required 3 faces of the cube of side L must be $\frac{Q}{8\epsilon_0}$.

Problem 2 – Conductors

Consider any closed surface anywhere inside the conductor (it may be arbitrarily small). Since the electric field is zero inside the conductor, the flux of the field will be zero too. Then by Gauss's law there can be no net charge inside the surface. Hence there can be no charge anywhere inside the surface. Charge can only reside on the outer surface of the conductor.

TAKE HOME POINTS: applications of Gauss's law

- You can simplify the calculations of your surface integral by using the symmetries of the vector fields and shapes of the surfaces. E.g. spherical surface with radially outward fields, or surfaces where the field is perpendicular to the surface normals.
- Imagine charges as the perpetual sources of flux, with $\frac{Q}{\epsilon_0}$ the amount of electric flux 'flowing' out of positive charges and 'flowing' into the negative charges. Gauss's law just says that the total flux out of a surface is the net flux produced by the charges inside it.

  $$\oiint_S \vec{E} \cdot d\vec{S} = Q_{enclosed}/\epsilon_0$$

  There is a similar law for magnetic fields: $\oiint_S \vec{B} \cdot d\vec{S} = 0$. This is because magnetic charges (the poles) always occur in negative and positive pairs.
- **In the future:** Gauss's law can also we written in a differential form $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, where $\vec{\nabla} \cdot \vec{E}$ is the divergence of electric field (kind of like flux per unit volume at a point) and ρ is the charge per unit volume at that point (charge density). The integral and differential forms are related by Divergence theorem, which we'll see in the future.