Solutions to Prelim 1 5b) *Intended* **and** *As Written*

Problem 5) (25 Points) Let

$$\begin{aligned} x = &\quad u^2 - v^2 \\ y = &\quad 2uv. \end{aligned}$$

Suppose $z = f(x, y)$ is a differentiable function.
5a) Express

$$\frac{\partial z}{\partial u}$$

in terms of

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad u, \text{ and } v.$$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = 2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}.$$

Similarly, not needed til 5b):

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (-2v)\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}.$$

5b Intended) Show that
This is a moderate difficulty problem similar to homework problem 38 from section 14.6

$$\frac{1}{4(u^2+v^2)}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

5b As Written) Show that
Because of the typo, even though it is doable, this is too hard a problem for people to do without practice. Consequently it was not counted in grading the prelim. **Sorry for the mistake!**

$$\frac{1}{4(u^2+v^2)}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

Solution to 5b) Intended: Using the formulas above in the 5a solution (including the z_v one)

$$\begin{aligned} \frac{1}{4(u^2+v^2)}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) = &\quad \frac{1}{4(u^2+v^2)}\left[\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right)^2\right. \\ &\quad \left. + \left(-2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right)^2\right] \end{aligned}$$

Since the cross term $4uv\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$ shows up twice on the right hand side with opposite signs, these terms cancel, and the right hand simplifies to

$$\frac{1}{4(u^2+v^2)}\left[(4u^2+4v^2)\left(\frac{\partial z}{\partial x}\right)^2 + (4u^2+4v^2)\left(\frac{\partial z}{\partial y}\right)^2\right]$$

which further simplifies to the desired

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

Solution to 5b) As Written: The most important additional idea needed here is to realize that the task (in terms of the chain rule) of differentiating (with respect to e.g. u) an expression like $\frac{\partial z}{\partial x}$ is just like the task of differentiating $z(x, y)$. So using the exact same argument as in the solution to 5a, we quickly have

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) &= 2u\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + 2v\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &&= 2u\frac{\partial^2 z}{\partial x^2} + 2v\frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right) &= (-2v)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + 2u\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &&= -2v\frac{\partial^2 z}{\partial x^2} + 2u\frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) &= 2u\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) + 2v\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) &&= 2u\frac{\partial^2 z}{\partial x \partial y} + 2v\frac{\partial^2 z}{\partial y^2} \\
\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right) &= (-2v)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) + 2u\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) &&= -2v\frac{\partial^2 z}{\partial x \partial y} + 2u\frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

Then using the solution to 5a) again,

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) &= \\
\frac{\partial}{\partial u}\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right) &= 2\frac{\partial z}{\partial x} + 2u\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\right) + 2v\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\right) \\
&= 2\frac{\partial z}{\partial x} + 2u\left(2u\frac{\partial^2 z}{\partial x^2} + 2v\frac{\partial^2 z}{\partial x \partial y}\right) \\
&\quad + 2v\left(2u\frac{\partial^2 z}{\partial x \partial y} + 2v\frac{\partial^2 z}{\partial y^2}\right)
\end{aligned}$$

which simplifies to

$$\frac{\partial^2 z}{\partial u^2} = 2\frac{\partial z}{\partial x} + 4u^2\frac{\partial^2 z}{\partial x^2} + 8uv\frac{\partial^2 z}{\partial x \partial y} + 4v^2\frac{\partial^2 z}{\partial y^2}.$$

Similarly one finds (in part because $\frac{\partial}{\partial v}\left((-2v)\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right)$ has a $-2\frac{\partial z}{\partial x}$ term) that

$$\frac{\partial^2 z}{\partial v^2} = -2\frac{\partial z}{\partial x} + 4v^2\frac{\partial^2 z}{\partial x^2} - 8uv\frac{\partial^2 z}{\partial x \partial y} + 4u^2\frac{\partial^2 z}{\partial y^2}.$$

Adding the two expressions quickly gives the desired result.