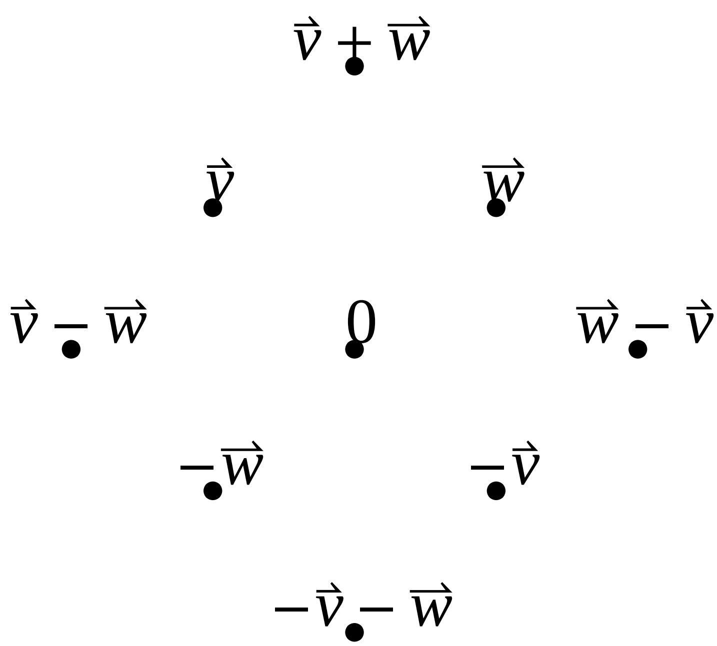
Math 2130 Homework 2: 13.1-13.4

Complete the following exercises on separate sheets of paper. Be sure to read over the presentablity guidelines (on the 2130 webpage) first.

(1) Write out the vector from the point $(1, 2, 3)$ to the origin in $\vec{i}, \vec{j}, \vec{k}$ format.

$-\vec{i} - 2\vec{j} - 3\vec{k}$

(2) Let $\vec{v} = (-1, 1)$ and $\vec{w} = (1, 1)$. In a single diagram, plot out the 9 points $a\vec{v} + b\vec{w}$ for $a = -1, 0,$ and 1 and $b = -1, 0,$ and 1. Clearly label each point (e.g. label the point $\vec{v} - \vec{w}$ as "$\vec{v} - \vec{w}$").



(3) Find the unit vector in the direction of $(1, 2, 3)$.

$\frac{(1,2,3)}{||(1,2,3)||} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$

(4) Describe when $||\vec{v} + \vec{w}|| = ||\vec{v}|| + ||\vec{w}||$.

You can work this out with geometric intuition, or with dot products. We're trying to solve when:

$$\sqrt{(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w})} = \sqrt{\vec{v} \cdot \vec{v}} + \sqrt{\vec{w} \cdot \vec{w}}.$$

Squaring both sides:

$$(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + 2\sqrt{\vec{v} \cdot \vec{v}}\sqrt{\vec{w} \cdot \vec{w}} + \vec{w} \cdot \vec{w}.$$

FOILing the left hand side:

$$\vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = \vec{v} \cdot \vec{v} + 2\sqrt{\vec{v} \cdot \vec{v}}\sqrt{\vec{w} \cdot \vec{w}} + \vec{w} \cdot \vec{w}.$$

Subtracting $\vec{v}\cdot\vec{v}$ and $\vec{w}\cdot\vec{w}$ from both sides:

$$2\vec{v}\cdot\vec{w} = 2\sqrt{\vec{v}\cdot\vec{v}}\sqrt{\vec{w}\cdot\vec{w}}.$$

Dividing both sides by 2, and rewriting in terms of vector lengths:

$$||\vec{v}||||\vec{w}||\cos\theta = ||\vec{v}||||\vec{w}||$$

Where θ is the angle between $\vec{v}$ and $\vec{w}$. As such we can conclude that either $\vec{v}$ or $\vec{w}$ has length 0, or $\theta = 0$. That is, $\vec{v}$ and $\vec{w}$ are in the same direction.

This matches with our geometric intuition: if $\vec{v}$ and $\vec{w}$ are in the same direction, their lengths add together. If $\vec{v}$ and $\vec{w}$ formed a triangle, the length of the third side ($\vec{v}+\vec{w}$) would be less than the sum of the lengths of the other two sides. If they were in exact opposite directions, the length would be the difference of the lengths of $\vec{v}$ and $\vec{w}$.

(5) Given a nonzero vector $\vec{v}$, for what values of a is $a\vec{v}$ a unit vector?

This works for $a = \pm\frac{1}{||\vec{v}||}$.

(6) Find the coordinates of the point of length 3 in two dimensions oriented at an angle of $\frac{\pi}{3}$ clockwise from the x-axis.

$(3\cos(-\frac{\pi}{3}), 3\sin(-\frac{\pi}{3})) = (\frac{3}{2}, -\frac{3\sqrt{3}}{2})$.

(7) If $\vec{v} = (1,2,3)$ and $\vec{w} = (1,0,1)$ compute $\vec{v}\cdot\vec{w}$ and $\vec{v}\times\vec{w}$.

$$\begin{aligned}
\vec{v}\cdot\vec{w} =& 1\cdot 1 + 2\cdot 0 + 3\cdot 1\\
=& 4\\
\vec{v}\times\vec{w} =& (\vec{i}+2\vec{j}+3\vec{k})\times(\vec{i}+\vec{k})\\
=& \vec{i}\times\vec{i} + 2\vec{j}\times\vec{i} + 3\vec{k}\times\vec{i} + \vec{i}\times\vec{k} + 2\vec{j}\times\vec{k} + 3\vec{k}\times\vec{k}\\
=& \vec{0} - 2\vec{k} + 3\vec{j} - \vec{j} + 2\vec{i} + \vec{0}\\
=& 2\vec{i} + 2\vec{j} - 2\vec{k}.
\end{aligned}$$

(8) If $\vec{v} = (1,2,3)$ and $\vec{w} = (1,0,1)$, find the perpendicular and parallel components of $\vec{v}$ in the direction of $\vec{w}$.

Turning $\vec{w}$ into a unit vector, which we will call $\vec{u}$, we get: $\vec{u} = \frac{\vec{w}}{||\vec{w}||} = \frac{1}{\sqrt{2}}(1,0,1)$. Using the projection formula from the textbook,

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= (\vec{v}\cdot\vec{u})\vec{u} \\
&= ((1,2,3)\cdot\frac{1}{\sqrt{2}}(1,0,1))\frac{1}{\sqrt{2}}(1,0,1) \\
&= \frac{1}{2}((1,2,3)\cdot(1,0,1))(1,0,1) \\
&= \frac{1}{2}\cdot 4(1,0,1) \\
&= (2,0,2) \\
\vec{v}_{\text{perpendicular}} &= \vec{v} - \vec{v}_{\text{parallel}} \\
&= (1,2,3)-(2,0,2) \\
&= (-1,2,1)
\end{aligned}$$

(9) If $\vec{v} = (1,0,1)$ and $\vec{w} = (0,1,1)$, find the angle between $\vec{v}$ and $\vec{w}$.

Recall the formula: $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||||\vec{w}||}$. As such:

$$\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

Or: $\theta = \pm\frac{\pi}{3}$. Either way, it's the same angle: $\frac{\pi}{3}$.

(10) Find a unit vector which is perpendicular to $(1,0,1)$ and $(0,1,1)$.

We can find a vector which is perpendicular to $(1,0,1)$ and $(0,1,1)$ by taking their cross product:

$$(\vec{i}+\vec{k})\times(\vec{j}+\vec{k}) = \vec{i}\times\vec{j}+\vec{i}\times\vec{k}+\vec{k}\times\vec{j}+\vec{k}\times\vec{k} = \vec{k}-\vec{j}-\vec{i}+\vec{0} = (-1,-1,1)$$

This vector has length $\sqrt{3}$, so dividing by that length we get $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Negative this vector is also a valid answer.

(11) If a and b are scalars and $\vec{v}$ and $\vec{w}$ are vectors, which of the following expressions make sense? Do they represent vectors or scalars?

(a) $a + b\vec{v}$ DNE
(b) $||\vec{v}|| + a$ Scalar
(c) $\frac{||\vec{v}||}{\vec{v}}$ DNE
(d) $(a+b)\vec{v}$ Vector
(e) $\vec{v} + b\vec{w}$ Vector
(f) $(\vec{v}\cdot\vec{w}) + a$ Scalar
(g) $(\vec{v}\cdot\vec{w}) + \vec{v}$ DNE
(h) $\vec{v}\cdot(\vec{w}+\vec{v})$ Scalar

(i) $\vec{v} \cdot (\vec{v} \cdot \vec{w})$ $\boxed{\text{DNE}}$
(j) $(\vec{v} \times \vec{w}) + a$ $\boxed{\text{DNE}}$
(k) $(\vec{v} \times \vec{w}) + \vec{v}$ $\boxed{\text{Vector}}$
(l) $\vec{v} \times (\vec{w} + \vec{v})$ $\boxed{\text{Vector}}$
(m) $\vec{v} \times (\vec{v} \times \vec{w})$ $\boxed{\text{Vector}}$

(12) If $\vec{v}$ and $\vec{w}$ are vectors, write $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w})$ in terms of the lengths of $\vec{v}$ and $\vec{w}$.

FOILing out, we get $\vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} - \vec{w} \cdot \vec{w}$. Since $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$, we're left with:

$$\vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{w} = ||\vec{v}||^2 - ||\vec{w}||^2.$$

(13) Write the equation of a plane through the point $(1, 2, 3)$ and perpendicular to the vector $(1, 0, 1)$.

$x + z = 4$.

(14) Use the algebraic properties of the cross product (not either of the definitions of the cross product) to compute $((\vec{i} + \vec{j}) \times \vec{j}) \times \vec{i}$ where $\vec{i}, \vec{j}$ and $\vec{k}$ are the standard basis vectors.

Distributing, we get that $(\vec{i}+\vec{j}) \times \vec{j} = \vec{i} \times \vec{j} + \vec{j} \times \vec{j}$. Since any vector crossed with itself is 0, this is just $\vec{i} \times \vec{j} = \vec{k}$. Crossing the result with $\vec{i}$, we get: $\vec{k} \times \vec{i} = \vec{j}$.

(15) If $\vec{v}$ and $\vec{w}$ are vectors in this page, and $\vec{v}$ is pointing to the top of the page, and $\vec{w}$ is pointing to the right side of this page, in what direction is $\vec{v} \times \vec{w}$ pointing? Does your answer change if $\vec{v}$ and $\vec{w}$ rotate a few degrees within the page?

In both cases $\vec{v} \times \vec{w}$ points directly into the page.

(16) Find the volume of the parallelpiped formed by the vectors $(1, 0, 1), (0, 1, 1)$, and $(1, 2, 0)$.

This is the absolute value of the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

. Using the diagonals trick, we get a determinant of:

$$1 \cdot 1 \cdot 0 + 0 \cdot 1 \cdot 1 + 1 \cdot 0 \cdot 2 - 1 \cdot 1 \cdot 1 - 2 \cdot 1 \cdot 1 - 0 \cdot 0 \cdot 0 = -3$$

Which has absolute value 3.