Math 2130 Homework 5: 14.6-14.7

(1) A point located at distance r from the origin with angle θ counter-clockwise from the x-axis has x coordinate $r\cos\theta$. Suppose r and θ are functions of t. Write out a chain rule for $\frac{dx}{dt}$. Suppose r is increasing at a constant rate of one unit per second, and θ is increasing at a constant rate of one unit per second (spiralling away from the origin). Use your chain rule to write $\frac{dx}{dt}$ as a function of r and θ. (Normally you would write it in terms of x and y.)

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt}.$$

We have that: $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial x}{\partial \theta} = -r\sin\theta$, and $\frac{dr}{dt} = 1$ and $\frac{d\theta}{dt} = 1$. Therefore:

$$\frac{dx}{dt} = \cos\theta - r\sin\theta.$$

(2) The temperature in a room at position x, y, z relative to a heat source is given by $H(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$. A fly is flying around in the helix $(\cos t, \sin t, t)$ for $-2\pi \leq t \leq 2\pi$. Work out, using the multivariable chain rule, the rate of change in temperature of the fly at $t = \pi$.

The relevant chain rule is:

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt}.$$

We can work out that

$$\begin{aligned}
\frac{\partial H}{\partial x} &= -\, x(x^2+y^2+z^2)^{-\frac{3}{2}} \\
\frac{\partial H}{\partial y} &= -\, y(x^2+y^2+z^2)^{-\frac{3}{2}} \\
\frac{\partial H}{\partial z} &= -\, z(x^2+y^2+z^2)^{-\frac{3}{2}} \\
\frac{dx}{dt} &= -\sin t \\
\frac{dy}{dt} &= \cos t \\
\frac{dz}{dt} &= 1
\end{aligned}$$

At $t = \pi$, $(x, y, z) = (-1, 0, \pi)$. Plugging this in, we have that:

$$\begin{aligned}\frac{\partial H}{\partial x}(-1,0,\pi) =&(1+\pi^2)^{-\frac{3}{2}}\\ \frac{\partial H}{\partial y}(-1,0,\pi) =&0\\ \frac{\partial H}{\partial z}(-1,0,\pi) =&-\pi(1+\pi^2)^{-\frac{3}{2}}\\ \frac{dx}{dt}(\pi) =&0\\ \frac{dy}{dt}(\pi) =&1\\ \frac{dz}{dt}(\pi) =&1\end{aligned}$$

Plugging everything in to our original equation, we get that

$$\frac{\partial H}{\partial t}|_{t=\pi} = -\pi(1+\pi^2)^{-\frac{3}{2}}.$$

(3) The temperature in a room at position x, y, z relative to a heat source is given by $H(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$. A fly is flying around in the helix $(\cos t, \sin t, t)$ for $-2\pi \leq t \leq 2\pi$. Work out, by plugging in the formulas for $x(t), y(t)$, and $z(t)$ into the equation for H to directly get H as a function of t, the rate of change in temperature of the fly at $t = \pi$.

Plugging in, we get:

$$H(t) = (\cos^2 t + \sin^2 t + t^2)^{-\frac{1}{2}} = (1+t^2)^{-\frac{1}{2}}.$$

Using the single variable calculus chain rule, we get:

$$\frac{dH}{dt} = -t(1+t^2)^{-\frac{3}{2}}$$

Which is $-\pi(1+\pi^2)^{-\frac{3}{2}}$, when we plug in $t = \pi$.

(4) Verify that $f_{xy} = f_{yx}$ for $f(x, y) = x^y$

$f_x = yx^{y-1}$. Then taking the partial derivative with respect to y, we get:

$$f_{xy} = x^{y-1} + yx^{y-1}\ln x.$$

Meanwhile, $f_y = x^y \ln x$. Taking the partial derivative with respect to x, we get:

$$f_{yx} = yx^{y-1}\ln x + x^y\frac{1}{x}.$$

As desired, these are equal.

(5) Use a quadratic approximation for $f(x, y) = x^y$ around $(x, y) = (3, 2)$ to approximate $3.02^{1.99}$.

$$\begin{aligned}
f(3,2) &= 9 \\
f_x(x,y) &= yx^{y-1} \\
f_x(3,2) &= 6 \\
f_y(x,y) &= x^y \ln(x) \\
f_y(3,2) &= 9\ln(3) \\
f_{xx}(x,y) &= y(y-1)x^{y-2} \\
f_{xx}(3,2) &= 2 \\
f_{xy}(x,y) &= x^{y-1} + yx^{y-1}\ln x \\
f_{xy}(3,2) &= 3 + 6\ln 3 \\
f_{yy}(x,y) &= x^y(\ln(x))^2 \\
f_{yy}(3,2) &= 9(\ln 3)^2
\end{aligned}$$

As such, the equation of our quadratic approximation is:

$$Q(x,y) = 9+6(x-3)+9\ln(3)(y-2)+(x-3)^2+(3+6\ln 3)(x-3)(y-2)+\frac{9}{2}(\ln(3))^2(y-2)^2.$$

Plugging in $(3.02, 1.99)$, we get:

$$Q(3.02,1.99) = 9+6(.02)+9\ln(3)(-.01)+(.02)^2+(3+6\ln 3)(.02)(-.01)+\frac{9}{2}(\ln(3))^2(-.01)^2$$

For comparison, this is $Q(3.02, 1.99) = 9.020149\ldots$. The exact value is $9.020151\ldots$.