Math 2130 Workshop: The Laplace and Wave Equations

The *Laplace Equation* describes heat distribution in regions after they have settled down to a steady state (the heat distribution isn't changing over time). In this case, the temperature at any location inside the region is the average of the temperature in a sphere around that point. If the temperature in the region at position (x, y, z) is given by $f(x, y, z)$, then:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Functions that satisfy this differential equation are called *harmonic*. Of course this holds if the whole region is a constant temperature; but there are other distributions that work as well (imagine a room being heated from the outside).

For each problem in this workshop, try to work out an intuition for why each function represents what they do.

1) Show that $f(x, y, z) = 2x + 2y$ is harmonic (a room being heated from the northeast corner and losing heat out the southwest corner).

2) Show that $f(x, y, z) = x^2 - y^2$ is harmonic (a room being heated from the east and west and losing heat from the north and south).

3) Show that $f(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ is harmonic except at the origin (a room being heated from a heat source at the center).

4) Show that $f(x, y, z) = e^{-2y} \cos 2x$ is harmonic.

The *Wave Equation* describes the propagation of waves: sound, light, and even waves on bodies of water. In the one-dimensional case (water waves in a trough or waves propagating down a string), if the intensity of the wave at position x and time t is given by $f(x,t)$, then:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

5) Show that $f(x,t) = \sin(x + c\,t)$ (a sinusoidal wave moving to the left) satisfies the wave equation.

6) Show that $f(x,t) = 5\sin(x + c\,t) - \cos(10\,x - 10\,c\,t)$ (a large sinusoidal wave moving to the left, with smaller, higher frequency waves moving to the right on top of them) satisfies the wave equation.

7) Show that $f(x,t) = \sin(c\,t)\cos(x)$ (a standing wave) satisfies the wave equation.

8) Show that $f(x,t) = g(x + c\,t)$ (an arbitrary configuration moving to the left) satisfies the wave equation.