HW 3 SOLUTIONS MATH 2210

RAKVI

1. Problem 1.7.6

Row reduce the augmented matrix for $Ax = 0$.

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $Ax = 0$ has only the trivial solution, so the columns of A are linearly independent.

2. Problem 1.7.10

a The vector v_3 is in Span$\{v_1, v_2\}$ if and only if the equation $x_1v_1 + x_2v_2 = v_3$ has a solution. To find out, row reduce $[v_1\ v_2\ v_3]$, considered as an augmented matrix

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}$$

Second row gives that v_3 is never in Span$\{v_1, v_2\}$.

b Row reduce the augmented matrix $[v_1\ v_2\ v_3\ 0]$:

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{v_1, v_2, v_3\}$ is a linearly dependent set for all h.

3. Problem 1.7.12

To study the linear dependence of three vectors, say v_1, v_2, v_3, row reduce the augmented matrix $[v_1\ v_2\ v_3\ 0]$:

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 8 & 0 \\ 0 & -5 & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

4. Problem 1.7.26

The columns must be linearly independent, by theorem 7, because the first column is not zero, the second column is not a multiple of the first, and the third column is not a linear combination of the preceding two columns. There should be a pivot in each column.

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5. Problem 1.7.28

There should be a pivot in each row. Since each pivot has to be in a different column, there are 5 pivot columns.

6. Problem 1.7.36

False. Here is a counterexample $v_1 = \{1,1,1,1\}$, $v_2 = \{2,2,2,2\}$, $v_3 = \{1,0,0,0\}$, $v_4 = \{4,4,4,4\}$.

7. 1.8.4

$$[A\,b] = \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x = \{-5, -3, 1\}$, unique solution

8. Problem 1.8.10

Solve $Ax = 0$. $$\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = x_3[-3, -2, 1, 0]$

9. Problem 1.8.12

Similar to some of the previous problems, we row reduce the augmented matrix $[A\ b]$ and it turns out that the system is inconsistent, so b is not in the range of the transformation $x \to Ax$.

10. Problem 1.8.20

Use the basic definition of Ax to construct A. Write $T(x) = x_1v_1 + x_2v_2 = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} x$, $A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$

11. Problem 1.8.24

Given any x in $\mathbb{R}^n$, there are constants $c_1, \ldots, c_p$ such that $x = c_1v_1 + \ldots + c_pv_p$, because $v_1, \ldots, v_p$ span $\mathbb{R}^n$. Then, from property (5) of a linear transformation, $T(x) = c_1T(v_1) + \ldots + c_pT(v_p) = c_10 + \ldots + c_p0 = 0$.

12. Problem 1.8.34

Let $x = c_1u + c_2v$ be the vector which is given to us in hint. Only thing left to prove now is that x is not zero. Suppose x is 0, then since we know that u and v are linearly independent, $c_1 = c_2 = 0$ which is a contradiction.

13. Problem 1.9.10

Applying the two transformations mentioned to e_1 we get $-e_2$ and to e_2 we get $-e_1$. So $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

14. Problem 1.9.22

Matrix associated to T is $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$

To solve $T(x) = [-1, 4, 9]$, row reduce the augmented matrix and we get $x = [5, 3]$.

15. Problem 1.9.24

- a False. Refer to paragraph preceding example 2.
- b True. Follows from theorem 10.
- c True. Refer to table 1.
- d False.
- e True. See example 5.

16. Problem 1.9.30

These are possible echelon forms:

$$\begin{bmatrix} \bullet & \star & \star & \star \\ 0 & \bullet & \star & \star \\ 0 & 0 & \bullet & \star \end{bmatrix}, \begin{bmatrix} \bullet & \star & \star & \star \\ 0 & \bullet & \star & \star \\ 0 & 0 & 0 & \bullet \end{bmatrix}, \begin{bmatrix} \bullet & \star & \star & \star \\ 0 & 0 & \bullet & \star \\ 0 & 0 & 0 & \bullet \end{bmatrix}, \begin{bmatrix} 0 & \bullet & \star & \star \\ 0 & 0 & \bullet & \star \\ 0 & 0 & 0 & \bullet \end{bmatrix}$$

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