## SOLUTIONS TO HOMEWORK 11

### Section 6.6 : Applications to linear models.

**6:** If the columns of $X$ are linearly dependent, then the same dependence relation would hold for the vectors in $\mathbb{R}^3$ formed from the top three entries of the column. In this case, the Vandermonde matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

would be noninvertible. However, it can be shown that since $x_1, x_2$ and $x_3$ are distinct, this matrix is invertible, which means that the columns of $X$ are, in fact, linearly independent. As in Exercise 5, Theorem 14 implies that there is only one least-squares solution for $\mathbf{y} = X\boldsymbol{\beta}$. One way to show that the above matrix is invertible is to show that its determinant is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

**14:** Write the design matrix as $X = \begin{bmatrix} \mathbf{1} & \boldsymbol{x} \end{bmatrix}$. Since the residual vector, $\boldsymbol{\epsilon} = \boldsymbol{y} - X\boldsymbol{\beta}$, is orthogonal to Col$X$, we have (using the notation shown just after Exercise 14)

$$\begin{aligned} 0 &= \mathbf{1} \cdot \boldsymbol{\epsilon} = \mathbf{1} \cdot (\boldsymbol{y} - X\boldsymbol{\beta}) = \mathbf{1}^T\boldsymbol{y} - \left(\mathbf{1}^T X\right)\hat{\boldsymbol{\beta}} \\ &= (y_1 + \cdots + y_n) - \begin{bmatrix} n & \Sigma x \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \\ &= \Sigma y - n\hat{\beta}_0 - \hat{\beta}_1 \Sigma x \end{aligned}$$

**20:** $\left\|X\hat{\boldsymbol{\beta}}\right\|^2 = \left(X\hat{\boldsymbol{\beta}}\right)^T \left(X\hat{\boldsymbol{\beta}}\right) = \hat{\boldsymbol{\beta}}^T X^T X \boldsymbol{\beta}\hat{} = \hat{\boldsymbol{\beta}}^T X^T \boldsymbol{y}$, because $\hat{\boldsymbol{\beta}}^T$ satisfies the normal equations: $X^T X\boldsymbol{\beta} = X^T\boldsymbol{y}$. Since $\left\|X\hat{\boldsymbol{\beta}}\right\|^2 = SS(R)$ and $\boldsymbol{y}^T\boldsymbol{y} = \|\boldsymbol{y}\|^2 = SS(T)$, Exercise 19 shows that

$$SS(E) = SS(T) - SS(R) = \boldsymbol{y}^T\boldsymbol{y} - \hat{\boldsymbol{\beta}}^T X^T \boldsymbol{y}.$$

### Section 7.1 : Diagonalization of Symmetric Matrices.

**16:** $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

**26:**

- **a:** False, by Theorem 2
- **b:** True. See the displayed equation in the paragraph before Theorem 2.
- **c:** False. An orthogonal matrix can be symmetric (and hence orthogonally diagonalizable), but not every orthogonal matrix is symmetric. The matrix $P$ in Example 2 is orthogonal but not symmetric.
- **d:** False, by Theorem 3(b).

**28:** Recall that

$$\begin{aligned} \left(B^T AB\right)^T &= B^T A^T B^{TT} \\ &= B^T AB \end{aligned}$$

The result about $B^TB$ is a special case for when $A = I$. $\left(B^TB\right)^T = B^TB^{TT} = B^TB$ , so $B^TB$ is symmetric.

**30:** If $A$ and $B$ are orthogonally diagonalizable, then $A$ and $B$ are symmetric, by Theorem 2. If $AB = BA$, then $(AB)^T = (BA)^T = A^TB^T = AB$. Hence $A$ is symmetric and hence orthogonally diagonalizable, by Theorem 2.

**Section 7.2 : Quadratic Forms.**

**10:** Indefinite; eigenvalues are $-7$ and 3. Change of variables; $\boldsymbol{x} = P\boldsymbol{y}$, with $P = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$. New quadratic form $-7y_1^2 + 3y_2^2$

**14:** Indefinite; eigenvalues are 1 and 4. Change of variables; $\boldsymbol{x} = P\boldsymbol{y}$, with $P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$. New quadratic form $-y_1^2 + 4y_2^2$

**22:**

- **a:** False. See the paragraph before Example 1.
- **b:** False. The matrix $P$ must be orthogonal and make $P^TAP$ diagonal. See the paragraph before Example 4.
- **c:** False. There are also "degenerate" cases: a single point, two intersecting lines, or no points at all. See the subsection "A Geometric View of Principal Axes."
- **d:** True. See the definition before Theorem 5.
- **e:** False, by Theorem 5(b). If $\mathbf{x}^TA\mathbf{x}$ has only negative values for $\mathbf{x} \neq 0$ , then $\mathbf{x}^TA\mathbf{x}$ is negative definite and has only negative eigenvalues.

**24:** If Det$A > 0$, then by Exercise 23, $\lambda_1\lambda_2 > 0$, so that $\lambda_1$ and $\lambda_2$ have the same sign; also, $ad = \text{Det } A + b^2 > 0$.

- **a:** If Det$A > 0$ and $a > 0$, then $d > 0$ because $ad > 0$. By Exercise 23, $\lambda_1 + \lambda_2 = a + d > 0$. Since $\lambda_1$ and $\lambda_2$ have the same sign, they are both positive. So $Q$ is positive definite, by Theorem 5.
- **b:** If Det $A > 0$ and $a < 0$, then $d < 0$ also. As in (a), we conclude that $\lambda_1$ and $\lambda_2$ are both negative, so $Q$ is negative definite.
- **c:** If Det $A > 0$, then by Exercise 23, $\lambda_1\lambda_2 < 0$, which shows that $\lambda_1$ and $\lambda_2$ have opposite signs. By Theorem 5, $Q$ is indefinite.