SOLUTIONS TO HOMEWORK 5

Section 4.3 Linearly independent sets; Bases.

4: Yes. See Example 5 of this section for an example of a justification.

14: Basis for Nul A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$ and the basis for Col A is given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}.$$

22:

a: False. The subspace spanned by the set must also coincide with H. See the definition of a basis.

b: True. This is true by the Spanning Set Theorem, applied to V instead of H. (V is nonzero because the spanning set uses nonzero vectors).

c: True. See the subsection "Two Views of a Basis."

d: False. See two paragraphs before Example 8.

e: False. See the warning after Theorem 6.

38: Writing $c_1 + c_2 \cos t + c_3 \cos^2 t + c_4 \cos^3 t + c_5 \cos^4 t + c_6 \cos^5 t + c_7 \cos^6 t = 0$ with $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ gives a 7×7 coefficient matrix A for the homogeneous systems $A\mathbf{c} = \mathbf{0}$. The matrix A is invertible, so the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution so $\{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t\}$ is a linearly independent set of functions.

Section 4.4 Coordinate Systems.

10: $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$

16:

a: True. See Example 2.

b: False. By definition, the coordinate mapping goes in the reverse direction.

c: True, when the plane passes through the origin, as in Example 7.

24: Given $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_{\mathcal{B}} = \mathbf{y}$. So, the coordinate mapping transforms $\mathbf{u}$ into $\mathbf{y}$. Since $\mathbf{y}$ was arbitrary, the coordinate mapping is onto.

26: $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_p$ if and only if there exist scalars $c_1, \cdots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p. \quad (0.1)$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_{\mathcal{B}} = c_1 [\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p [\mathbf{u}_p]_{\mathcal{B}} . \tag{0.2}$$

Conversely, (0.1) implies (0.2) because the coordinate mapping is one-to-one. Thus $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_p$ if and only if (0.2) holds for some $c_1, \cdots, c_p$, which is equivalent to saying that $[\mathbf{w}]_{\mathcal{B}}$ is a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}, \cdots, [\mathbf{u}_p]_{\mathcal{B}}$.

Section 4.5 The dimension of a Vector Space.

14: $3, 3$

20:

a: False. The only subspaces of $\mathbb{R}^3$ are listed in Example 1. $\mathbb{R}^2$ is not even a subspace of $\mathbb{R}^3$, because vectors in $\mathbb{R}^3$ have three coordinates. Review Example 8 in Section 4.1.

b: False. The number of free variables equals the dimension of Nul A. See the box before Example 5.

c: False. Read carefully the definition before Example 1. Not being spanned by a finite set is not the same as being spanned by an infinite set. The space $\mathbb{R}^2$ is finite-dimensional, yet it is spanned by the infinite set S of all vectors of the form (x, y), where x, y are integers. (Of course the two vectors $(1, 0)$ and $(0, 1)$ in S are enough to span $\mathbb{R}^2$.)

d: False. S must have exactly n elements to be a basis of V. See the Basis Theorem.

e: True. See Example 4.

Section 4.6 Rank.

10: 1

18:

a: False. Review the warning after the proof of Theorem 6 in Section 4.3.

b: False. See the warning after Example 2. For instance, a row interchange usually changes dependence relations among rows.

c: True. See the remark in the proof of the Rank Theorem.

d: True. This fact was noted in the paragraph before Example 4. It also follows from the fact that the rows of a matrix -say, A^T-are the columns of its transpose, and $A^{TT} = A$.

e: True. See Theorem 13.

20: No. The presence of two free variables indicates that the null space of the coefficient matrix A is two-dimensional. Since there are eight unknowns, A has eight columns and therefore must have rank 6, by the Rank Theorem. Since there are only six equations, A has six rows, and Col A is a subspace of $\mathbb{R}^6$. Since Rank $A = 6$, we conclude that Col $A = \mathbb{R}^6$, which means that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

28:

a: dim Row A = dim Col A =Rank A, by the Rank Theorem. So part (a) follows from the second part of that theorem.

b: Apply part (a) with A replaced by A^T and use the fact that Row A^T is just Col A.