SOLUTIONS TO HOMEWORK 8

Section 5.6 Discrete Dynamical Systems.

2: $\mathbf{x}_k = 2 \cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + 1 \cdot \left(\frac{4}{5}\right)^k \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + 2 \cdot \left(\frac{3}{5}\right)^k \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$ and for k sufficiently large, $\mathbf{x}_k \approx 2 \cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

8: Saddle point (because one or more eigenvalues are greater than 1, and one or more eigenvalues are less than 1, in magnitude); direction of greatest repulsion: the line through $(0,0,0)$ and $(1,0,-3)$; direction of greatest attraction: the line through $(0,0,0)$ and $(-3,-3,7)$.

12: Saddle point; eigenvalues: 1.1, 0.8; greatest repulsion: the line through $(0,0)$ and $(1,1)$; greatest attraction: the line through $(0,0)$ and $(2,1)$.

Section 5.7 Applications to Differential Equations.

6: $-\begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$. The origin is an attractor. The direction of greatest attraction is the line through $(0,0)$ and $(2,3)$.

8: Set $P = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A = PDP^{-1}$. Substituting $\mathbf{x} = P\mathbf{y}$ into $\mathbf{x}' = A\mathbf{x}$, we have $\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y})$ so $P\mathbf{y}' = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$. Left multiplication by P^{-1} gives $\mathbf{y}' = D\mathbf{y}$, or $\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

12: (complex): $c_1 \begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{(-1+2i)t} + c_2 \begin{bmatrix} 3+i \\ 2 \end{bmatrix} e^{(-1-2i)t}$

(real): $c_1 \begin{bmatrix} 3\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3\sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix} e^{-t}$. The trajectories spiral in, toward the origin.

Section 6.1 Inner Product, Length, and Orthogonality.

10: $\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$

20:

a: True. See Example 1 and Theorem 1(a).
b: False. The absolute value is missing. See the box before Example 2.
c: True, by definition of orthogonal complement.
d: True, by Pythagorean Theorem.
e: True, by Theorem 3.

24: $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$
$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}-2\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$
When $\|\mathbf{u}+\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2$ are added, the $\mathbf{u}\cdot\mathbf{v}$ terms cancel, and the result is $2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$.

28: An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u},\mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u}+c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y}=0$ and $\mathbf{v}\cdot\mathbf{y}=0$. By linearity of the inner product (Theorem 1(b) and 1(c)),

$$\mathbf{w}\cdot\mathbf{y} = (c_1\mathbf{u}+c_2\mathbf{v})\cdot\mathbf{y} = c_1\mathbf{u}\cdot\mathbf{y}+c_2\mathbf{v}\cdot\mathbf{y} = c_1 0+c_2 0 = 0$$

30:

a: If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u} = c(\mathbf{z}\cdot\mathbf{u}) = c0 = 0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.

b: Take any $\mathbf{z_1},\mathbf{z_2}$ in $W^\perp$. Then for any $\mathbf{u}$ in W, $(\mathbf{z_1}+\mathbf{z_2})\cdot\mathbf{u} = \mathbf{z_1}\cdot\mathbf{u}+\mathbf{z_2}\cdot\mathbf{u} = 0+0 = 0$, which shows that $\mathbf{z_1}+\mathbf{z_2}$ is in $W^\perp$.

c: Obviously $\mathbf{0}$ is in $W^\perp$ because $\mathbf{0}$ is orthogonal to every vector. This fact, together with (a) and (b) show that $W^\perp$ is a subspace.