WHAT TO DO TILL THE TOPOLOGIST COMES
(MATHEMATICAL ACTIVITIES FOR CHILDREN)

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Here are some mathematical activities to do with children that are enjoyable, easy to do, thought provoking, and subversively educational (no one needs to mention who is learning what). They can be done with any age group from pre-school to high school (if presented appropriately), for large groups or small groups, in a classroom or a social setting (camps, parties, family gatherings, etc.) These are participatory activities, so you will need enough supplies for everyone. The children might want to work in groups of two or three, which is fine provided no child is relegated to an inactive role.

Supplies required: string (heavier is better, but it should be easy to tie), paper, scissors, scotch tape, crayons or markers, donuts.

As a come–on you can inform the children that at the end of the experiments the donuts will have to be eaten, for a good cause. In addition, you will need one each of the following props:

Props: ball, balloon (blown up), paper plate, inner tube from paper towel roll, tee–shirt, rubber band, mirror, plastic transparency sheet.

If there is no blackboard in the room then you will need a large sheet of paper to draw diagrams so that everyone can see.

The topics illustrated by these activities are curves, surfaces, boundaries, knots and links, which belong to the branch of mathematics called topology, which just means qualitative geometry (geometric properties that are independent of exact measurements). Depending on the age group and interests of the children, you may take the opportunity to discuss these ideas in the course of the activities. The activities can be compressed into a single one hour session, or spread over several sessions, with some exercises assigned for homework. Some of the more complicated constructions can be omitted, and there are many suggestions for problems and projects for those who are more ambitious.

A leading question for these activities is the following: do you think a surface can have a knotted loop as its boundary?
§1 Curves, loops and knots

A piece of string represents a curve. The curve can have two ends, or the ends can be joined together to form a loop. The rubber band is a familiar example of a loop. Can you think of other examples? Cut off a short piece of string and form a loop by holding the two ends together. Now make a knotted loop. The simplest kind is a trefoil knot, made by passing the string over, under and then over itself

![Trefoil knot diagram]

and then tying the two loose ends together.

![Tied knot diagram]

It should be emphasized that the knot we use to tie the ends together is not part of the trefoil knot, but merely a necessary error in construction if we are to represent the ideal curve by a real object. (For older age groups you might start a discussion about other sorts of inaccuracies, such as the fact that the string is not a true line, paper is not a true surface, etc.) To emphasize this you might suggest using the scissors to trim the edges off the joining knot.

Now help everyone make their own trefoil knot. Draw the diagram on the board, let them watch you do it, and emphasize verbally the over—under—over sequence. Some children will have difficulty with this, so it is important to give encouraging assistance. Better still, get the children who have done it successfully to help the others.

Once everyone has a completed trefoil knot, you can explain that the true nature of its knottedness is the impossibility of transforming it into the unknotted loop (rubber band) configuration without cheating (cutting, or untying the join knot). Let everyone manipulate the trefoil knot in space to get an intuitive feeling for this fact. Probably your most difficult task is to convince them that it is not obvious that the trefoil knot cannot be unknotted (have fun with the tongue-twister). One way to do this is to build an ordinary loop of string of the same length, then separately tangle up both loops, and ask who can tell which is which. In fact you can make a simple game out of this. One child,
the **tangler**, gets to tangle up a simple loop and a trefoil knot. The tangler’s partner, the **guesser**, has to guess which is which. The issues is resolved by untangling the two loops. (For the sake of sanity in untangling, tight knots must be prohibited in tangled loops!)

Untangle all the trefoils and help everyone place them down on a table in standard position, so that they form an approximately symmetric clover leaf configuration

![Clover Leaf Configuration](image)

Are they all the same? With a little luck, you will be able to find examples of both left and right-handed trefoil knots (if not, quickly build the missing example).

![Left and Right Trefoil Knots](image)

The distinction is subtle: as you move across from the right does the string first pass under or first pass over itself? (There is really not much point in deciding which one to label left-handed). Have everyone rotate the knot and observe that the under or over passing stays the same. You cannot transform one trefoil knot into the other by rotating it. What happens if you flip it over? Ask the children to guess before they actually do it. (For those who guess wrong, make sure the message is “wow, isn’t this interesting” and not “uh–oh, something is wrong with me.”) Let them manipulate the two kinds of trefoil knots to convince themselves that there is no way to transform one into the other in space.

Then bring out the mirror. Look at one trefoil in the mirror and compare it with the other. This might be a good time to have a discussion of mirror reflection and handedness in general. What does your right hand look like in a mirror? Do you see yourself as others see you, or as you appear in a photo? What happens to writing in a mirror image?

For an older age group you might want to discuss the difference between intrinsic and extrinsic geometric consideration. The distinction between the simple loop and the trefoil knot has to do with extrinsic geometry: the way they sit in space. A blind ant crawling around the two would not be able to tell the difference (you have to imagine the two
suspended in space with no touching of crossing parts of the trefoil knot). The ant would have the choice of moving in either of two directions, and would eventually come back to its starting point (presumably the ant would be able to “smell” this fact). If the ant counted steps to measure distances, the ant would be able to distinguish the lengths of loops. Of course the ant would always be able to distinguish a loop from a two-ended segment because at the endpoints the ant would only be able to move in one direction.

**Problems and Projects:**
1) Construct more complicated knotted loops.
2) Can you construct a knot that is the same as its mirror image?
3) Construct two trefoil knots, but instead of closing the loops with a knot, join them together

![Diagram of connected trefoil knots]

There are several different ways you can do this, depending on the handedness of the knots and which ends you join. How many different knots do you obtain? What are their mirror images?
4) Make knots using garden hose instead of string. The two ends can be joined cleanly by screwing the hose connectors together.
5) Learn to draw diagrams of knots (indicating clearly overpasses and underpasses), and how to construct knots from the diagrams. This can get made into a communication game. One person makes a knot and draws a diagram of it. The second takes the diagram only and reconstructs the knot. If both have done the job properly, then the two knots will be the same.
6) Take any knot diagram and interchange all overpasses and underpasses. Is the resulting knot always the mirror image of the first knot?

**§2 Links**
Two simple loops may be separate (unlinked) or linked together. The simplest link is just

![Diagram of a simple link]

You can create such links with string, or more simply with the first two fingers of each hand
Is the mirror image the same or different?

You can make more complicated links using two or more loops (unknotted or knotted). But here is an interesting challenge: can you create a link with three loops which cannot be separated, yet if any one of the loops is removed then the remaining two loops are unlinked? This is a little hard to explain in words, but if you can manage it you can ask for opinions before you show the construction.

Have everyone cut three equal size pieces of string to make three loops. Tie two of them in loops (unknotted and unlinked), and place one on top of the other:

These two loops will not be linked together, regardless of what the third string does. Now take the third string, and beginning on the same side as the upper loop, pass it over, under over and under the successive strings.
and then tie the third loop closed.

The resulting link is the famous 3–ring sign.

By playing with the 3–ring sign it should become clear that the three loops cannot be separated. Nevertheless, if any one of the loops is cut, the other two fall apart. This can be demonstrated with a pair of scissors, but before doing this it should be possible to work through a conceptual “Proof” as follows. It is clear that cutting the third loop will restore the first two loops to their original unlinked state. But all three loops enter into the 3–ring sign symmetrically. This can be seen convincingly by rotating the 3–ring sign. It will quickly become impossible to tell which is the third loop.

The 3–ring sign can also be created by a group of three children. Have two of the children fold their hands to create loops out of their arms, and have them stand facing each other, with arms held forward (one child’s arms will rest on top of the others’).

Now the third child stands to the side, and places one arm over the topmost arm and the other arm under the bottom arm.
The third child completes the loop by passing the arm that went over, under, and the arm that went under, over, the next arms encountered, and then folds hands.

This does not really require any special flexibility or double–jointedness to achieve, but it does require concentration to get it right. The important idea to keep in mind is to alternate the under and over crossing. When the three children are linked, they can feel directly the fact that they cannot separate (without unfolding hands). Then the game is to have just one of the children let go, and the other two are magically unlinked.

Problems and Projects:
1) Is the mirror image of the 3–ring sign the same or different?
2) Make a trefoil knot with three children linking arms. Then switch to the other handed trefoil knot. Practice switching back and forth rapidly.
3) Experiment with different links involving four loops. Can you create a 4–ring sign which is linked, but which becomes completely unlinked if you cut any one loop?
4) Form a link with two loops with two twists

and its mirror image. Are they the same? Why is the answer different than in problem 1?

§3 Surfaces and boundaries

Begin by examining examples of surfaces from everyday life, and their boundaries. The paper plate has the circle as its boundary. Flex the paper plate to show that different “equivalent” loops are the boundaries of different “equivalent” surfaces. It is not necessary
to emphasize this too much, but with an older age group you can bring in a discussion of the idea (on an intuitive level) that if the paper plate were made of easily stretchable rubber, then any transformation of the boundary loop that was allowed in the tangling games could be accomplished with the boundary of the plate. Thus any simple loop in space is the boundary of a surface.

Next look at the inner tube from a roll of paper towels, and ask what its boundary is. Emphasize that the two loops are unlinked. Ask if anyone thinks a surface can have two linked loops as boundary. You can tease them a little with this, suggesting that it is very unlikely since the linked loops pass through each other and a surface can’t do this. The next example is the tee-shirt with its four loop boundary.

The question may arise as to what exactly is a boundary. If it doesn’t you might want to raise it yourself. Keep the discussion on a intuitive level, mentioning the old idea that you might fall off the edge of the earth. Most children will easily grasp the intuitive idea of “boundary”, but may not initially associate it with that word. If necessary you should mention that we are only going to deal with smooth surfaces and smooth boundaries, so that if a piece of paper has corners we will imagine that they are slightly rounded off.

Mention the analogy of the boundary of a surface and the ends of a piece of string. For an older audience you can discuss the dimensions involved, the 2-D surface having a 1-D boundary, and the 1-D curve having a 0-D ends. Of course a loop is a curve without ends, which leads naturally to the next question: are there surfaces without boundaries? Make sure you get some answers before bringing out the ball, balloon and donut. The point of the balloon is that we are thinking of the surface of the ball and donut, not the interior. An ant crawling on the ball and balloon would not be able to tell the difference. You can liken this to our experience on the surface of the earth... how do we really know what is down there? The surface of the donut is called a torus; you might suggest that the children think of the sugar (or other coating) on a sugar donut. At this point you can have a discussion of other surfaces that they can think of from their experience.

Now construct some more interesting surfaces out of paper. Have everyone cut a rectangular strip of paper and tape the short ends together to form an ordinary band.

See if you can get them to tell you that it is really the same surfaces as the inner tube from the roll of paper towels. Next have everyone make a Möbius band by giving one end a half twist before taping
One strategy for doing this correctly is to draw arrow marks in opposite directions on the ends

and making sure they point in the same direction when taping the ends together. Generally speaking, it will be easier to manipulate the paper if the rectangle is considerably longer than it is high.

What is the boundary of the Möbius band? Make sure that everyone is convinced that it is a single loop. If necessary, have the doubters color the edge with a marker. Is the loop knotted? If necessary, have the children wrap a string in the shape of the boundary (if the Möbius band sits on the table you can carefully imitate the boundary with string, being careful to do the exact same over and under crossings),

tie the ends together, and then unwrap the loop.

Now you can suggest a better method for deciding such questions. For the sake of
clarity, use a skinny band whose boundary has been colored. Take a scissors and cut the band along the middle

all the way around. You may have to fold the band slightly to get started, and it is important that you use a generous amount of tape when constructing the band so that the joint doesn't fall apart when you cut it. When you complete the cut, the resulting very skinny band is to be "interpreted" as a piece of string representing the boundary curve, which is now clearly seen to be an unknotted loop. Repeat the process for an ordinary band, with the result being two unlinked loops.

Depending upon the audience, you may want to give more or less emphasis on the conceptual switch whereby the whole band starts out "representing" the 2-D surface, but after being cut "represents" the 1-D boundary curve. You can point out that this means we are "ignoring" the 2-D aspects of the cut band, such as the way it twists in space. You could also briefly discuss the idea that nothing in the real world exactly matches our mathematical ideas. The paper is not infinitely thin, the string is not really one-dimensional. Nevertheless, the mathematical ideas are often useful in understanding the real world, when the correspondence is very crude, as it is with the cut band and the boundary curve.

Now suggest doing the same experiment on a band with two half-twists. If necessary, mark arrows on the short ends, but this time in the same direction,

![Arrows on ends of band]

and they should be lined up before taping the ends. Everyone should be sure that the band has two loops for boundary, and see if anyone is willing to guess that the loops are linked, before taking the scissors and cutting.

Well, if two is beautiful, three should be even better. You might try to coax this suggestion from the participants. It is a little harder to accomplish the three half-twists, so you might recommend even longer and skinnier rectangles to start with (of course not too skinny, so you can still cut down the middle). The arrow marks should again go in opposite directions, as with the original Möbius band, and the resulting band has again only a single loop boundary. Is it knotted? The scissors give the proof.

A side issue that can be explored at this time is the distinction between one-sided and two-sided surfaces. At first this idea seem paradoxical, since a piece of paper obviously has two sides. So you have to explain that there is a difference between "local" and "global". Every surface is locally two-sided, but not necessarily globally. The question can be put as follows: can you color one side of the surface consistently over the whole surface. If you color the original rectangular strip on one side, then the ordinary band, or the two
half-twist band, maintains this consistency when you tape, but not the Möbius band or the three half-twist band. Imagine trying to paint one side of a conveyor belt if a half twist were concealed on the underside? An ant crawling on a Möbius band could return to its original starting point but upside down.

The issue is really more subtle, because the mathematical idea of a surfaces does not really have sides the way a piece of paper does — the sides arise from the fact that the paper has some small thickness, and so deviates from the idea of a surface which would have no thickness. The question is whether or not you can consistently define left-handedness and right-handedness on the surface; the mathematical term is orientation. On a two-sided surface you can do this in one of two ways (drawing a left hand on one of the two sides)

while on a one-sided surface this is impossible. You can illustrate this by constructing an ordinary band and a Möbius band out of transparent plastic. Cut out a left hand shape on a piece of paper, and slide the left hand along the two surfaces. On the two-sided band, the left hand always looks like a left hand when viewed from the same side, but on the Möbius band it comes back as a right hand when it makes one trip around the band (of course it also has moved to the “other side.”)

Problems and Projects: 1) If a band is given $n$ half-twists, will the boundary consist of one or two loops? Will the surface have one or two sides? Can you explain why the two answers are the same? (Think about moving the left hand around the band, keeping the thumb pointed to the same edge).
2) Experiment with cutting down the middle a band with 4 and 5 half-twists. Can you guess what the general pattern is? Can you give an explanation? Try making careful diagrams of the boundary.
3) What does the Möbius band look like in a mirror? Construct both left – and right-handed Möbius bands and check that they cannot be transformed into each other (be sure to test what happens when you turn them inside-out).
4) From the point of view of intrinsic geometry (a blind ant crawling on the surface), how many distinct types of bands are there?
5) Reconsider the knotted garden hose from the point of view of the surface of the hose. Explain why this surface can be considered as a knotted torus.
6) The boundary of a smooth solid is a smooth surface. This surface is always two-sided. Can you give a plausible reason? This surface has no boundary. Can you give a plausible reason?

§4 Seifert Surfaces

Having constructed surfaces whose boundary consists of a pair of linked loops, or a
trefoil knot, we are tempted to believe that anything is possible — the 3-ring sign, and more complicated knots and links. This is in fact true: any finite set of linked knots is the boundary of a surface. Such surfaces are called Seifert surfaces (named after a German topologist who first discovered this fact). Moreover, it is possible to construct the surface to be two-sided. Remember that the three half-twist band whose boundary was a trefoil knot was one-sided, so we have not yet constructed the Seifert surface for a trefoil knot. In this section we will describe the construction of the Seifert surface for the trefoil knot and the 3-ring sign. These constructions are more difficult than the earlier ones, and take more time. You might suggest that the children work in pairs. To save time you can assign some of the preliminary template cutting as homework. You can omit these constructions altogether, or just do the first one.

To begin, have every group cut out two identical 3-armed starfish.

![Starfish Diagram]

The exact shape is not too important, as long as the straight edges of the three arms are roughly equal in length, since these will be taped together. Color one side of each starfish. Put one starfish on the table, and have one child hold the other starfish a few inches over it. Both starfish should have the colored side up. Observe that if the arms are joined to each other (each of the three arms of the lower starfish to the corresponding arm of the upper starfish) **without** any twists, then the resulting surface would have three unlinked loops as its boundary.

![Starfish with loops Diagram]
This is not what we want. Instead, have the child not holding the upper starfish give each arm of the lower starfish a half twist before taping it to the arm of the starfish held above it. This will make the colored sides match up.

This operation must be repeated three times, with the twist going in the same direction each time. This takes a certain amount of thought and dexterity. Putting a weighted object on the bottom starfish to hold it in place may be helpful. The arms of the bottom starfish can be twisted in advance in a consistent manner to enforce the uniform direction of the half twist. Longer armed starfish are easier to work with, but if the arms are not too long the surface will hold its shape better.

To see that the surface has a single loop for a boundary, take a marker and trace along the boundary on the uncolored side. The surface is clearly two-sided because one side is already consistently colored. Notice that this is different than the band surfaces, where all the two-sided bands had two loops for a boundary. To see that the boundary loop is indeed a trefoil knot you can draw a careful diagram of the boundary, but probably the scissors test will be more convincing (although it may seem a shame to destroy such a carefully crafted work). The cutting process here is a little more complicated than in the case of the bands. It has to be kept in mind that we are actually cutting out the boundary curve, not cutting the surface in half. When we get through there will be triangular portions of the center of each starfish that will be discarded because they contain no piece of the boundary edge. Since the boundary has been marked this will take care of itself. Start cutting in the middle of an arm and continue following the boundary curve (at some point, as the arm thickens, you will have to choose to follow one or the other sides of the arm). The cuts on each starfish look like
and the shaded triangle simply drops away. The boundary of the lower starfish (with the half twists) looks like

Attaching this to the boundary of the upper starfish gives us

which is clearly a trefoil knot.

To construct the second example cut out two more 3-armed starfish and one 6-armed starfish of about the same size.

Color one side of each starfish. Twist the arms of one of the 3-armed starfish as before, and also twist every other arm of the 6-armed starfish in the same direction (while both starfish are held colored side up).

The construction proceeds in two stages. To do the first stage, put the twisted 3-armed starfish down on the table, colored side up, and hold the 6-armed starfish a few inches above it, also colored side up, and rotate it so that its three untwisted arms are directly above the twisted arms of the lower starfish. If you imagine that the twisted arms of the 6-
armed starfish are not there, then you have the identical configuration of the first example. **Do exactly what you did before.** Give the twisted arms of the bottom starfish a half twist in the indicated direction and tape them to the three untwisted arms of the 6–armed starfish above it. The colors should match. The first stage is complete.

The second stage is similar to the first stage. Leave the half completed surface where it is, and bring the last 3–armed starfish a few inches over it, colored side up, with its arms directly over the as yet unused three twisted arms of the 6–armed starfish. We are back in the familiar situation, and we do the same thing as before. We give each of the twisted arms a half twist in the indicated direction and tape them to the arms of the overhead starfish. The colors should match. This completes stage two.

The completed surface has the 6–armed starfish in the middle, with every other arm going up and down to join the 3–armed starfish. Each connection has a half twist, all going in consistent directions. It is a two–sided surface with one side consistently colored.

What is the boundary like? Take three different colored markers, and start to trace the boundary (on the uncolored side) in the first color. Start at the bottom starfish. You will go up upstairs to the middle starfish, then up again to the top one, then down to the middle one, then down again to the bottom one. At each “floor” you will rotate in a consistent direction (1/3 around on the top and bottom floors, and 1/6 around, twice, on the middle floor) and you will return to your starting point, completing one of the loops of the boundary — provided the construction was done correctly. This does not use up the whole boundary. Take a different color and repeat the process, starting anywhere on the boundary of the bottom floor that has not yet been colored. Finally, do it once more time in a third color. This shows the boundary consists of three loops

(one boundary loop, 2=3, 4=5, 6=7, 8=1)

In fact the loops are linked in the 3–ring sign. If this is hard to see, you can take out the scissors and sacrifice beauty for truth.

**Problems and Projects:**
1) Construct two versions of the first example, making the half twists in different directions. Do you obtain the two versions of the trefoil knot as boundary?
2) Explore systematically what happens in the first construction if you vary the twists. Suppose not all half twists are in the same direction? What if you give each arm two half twists? How many possibilities can you think of that involve 0, 1 or 2 half twists for each of the arms (also allowing different directions)? What if you interchange two arms?
3) Explore other variations on the theme, using starfish with different numbers of arms. If in the second example the top and bottom starfish had 6 arms, you could join the 3 unused
arms of the top and bottom starfish together (this will be quite tricky to accomplish, and will require lifting the whole construction off the table). Then the three starfish will be joined cyclically, with no top, bottom or middle.

4) Imagine three round balloons, one inside the other, inside the third. This gives a configuration of three surfaces without boundary. Explain why there cannot be any solid which has this as its boundary.

5) Given any two surfaces, you can join them together by cutting small holes (anywhere) in each and attaching a tube between them

The resulting surface has for boundary the two boundaries of the initial surfaces which are not linked. Work out this construction with some of your models.

§5 Curves and Surfaces

So far we have worked with curves in space. What about curves on surfaces? You can draw a simple loop on a piece of paper (a plane),

but if you insist the curve never cross itself, every loop is unknotted. If the curve were made out of rubber, you could transform it to a circle, without ever having it cross itself. The idea of "No crossing" can be justified by the fact that the mathematical idea of a surface involves no thickness, so there can be no over or under crossings. The diagrams of knots and links deal with this problem by inserting a break in the curve to indicate an undercrossing.

Since nothing interesting happens in a plane, we have to look at more complicated surfaces. What about a sphere? Take out the blown-up balloon and ask if anyone thinks it is possible to draw a loop (without crossings) on the balloon that is knotted. There are really two questions here: 1) can the loop be untangled moving on the surface of the balloon?, 2) can the loop be untangled moving through space? If someone brings up the question you can try to get across the point that 1) is harder than 2), because the
untangling on the surface is also an untangling in space, but moving through space allows more movement.

Have someone draw a complicated loop on the balloon that might be knotted (make sure it is a loop and has no crossings). With some luck it will not pass through the tied end of the balloon.

Untie the end and let the air out (this is sometimes very hard to do, so you might just talk about doing it). The balloon could then be stretched out to lie flat on a piece of paper (again this is rather hard to do).

Since every loop on a piece of paper is unknotted, the loop on the balloon can be untangled by moving it on the plane of the paper. But the same untangling motion, carried out on the blown-up balloon, would show that the original loop was unknotted in the sense of question 1). So no matter how we look at it, there are no knots on a sphere. If someone asks what you would do if the original curve passed through the tied end of the balloon, find someone else to suggest moving it slightly.

To get some knotted loops on surfaces, bring out the donuts. What kind of loops can be drawn on a torus? The first thing to point out is that even unknotted loops are interesting. Ask for suggestions for different ways to draw simple loops on a torus. Eventually you should get the following three pictures:

1) a loop not surrounding anything,
2) a loop surrounding the donut,
3) a loop surround the hole.

It should be clear that these are all distinct in the sense that they cannot be transformed into each other by moving around on the surface of the donut, although they are equivalent in space.
Since it is difficult to draw pictures on a donut, you can suggest the following pictorial representation:

Imagine that the rectangle is made of rubber. You could tape together the two long edges marked with single arrows so that the direction of the arrows matches. The result would be the same as the inner tube from the roll of paper towels, but the ends of the tube are marked with double arrows rotating in the same direction.

Now imagine wrapping the rubber tube around and taping together the two remaining ends with the double arrows pointing in the same direction. The result would be a torus. Without doing any of these steps, the rectangle “represents” the torus in the following sense: every point inside the rectangle corresponds to a unique point on the finished torus, and points on the edges of the rectangle correspond to points on the torus in a non–unique fashion. For example, points on the long edges directly above each other

represent the same point on the torus because they get taped together in the first step of the construction. Similarly, points on the short edges directly across from each other

represent the same point on the torus because they get taped together in the second step of the construction. You can play a little game to make sure everyone understands this identification. Have one child mark a point on one of the boundary edges, and have another child mark the point on the opposite edge that represents the same point on the torus. Be sure to discuss the quadruple identification of the four corners. It should also be emphasized that points on the torus corresponding to edges in the picture are not in any
way “special.” From the point of view of the torus they are no different from any other points.

Using this “map” of the torus, we can draw the different kinds of curves on the torus. These will look like curves on the map, with the understanding that when the curve hits an edge it can “jump over” to the point on the opposite edge that represents the same point on the torus. These “breaks” in the curve on the map do not correspond to breaks in the curve on the torus. This means that loops on the torus may not necessarily look like loops on the map.

Now have the children try to figure out how to represent the three types of simple loops on the donut on the map. Eventually they should discover the three pictures

![Diagram of three types of simple loops on the torus.](image)

Make sure that they understand that the second picture gives a loop, but

![Diagram of the second type of loop.](image)

is not a loop. You could also suggest some verbal descriptions: the second loop goes once around the short way, while the third loop goes once around the long way.

Now we can draw pictures of more complicated loops. A diagonal

![Diagram of a diagonal loop.](image)

represents a loop that goes one around the short way and once around the long way (at the same time). Have the children try to visualize what is happening on the donut, and wrap a string around the donut to illustrate it. You might ask them to compare the diagonal loop with this one.

![Diagram of the comparison.](image)
Or ask if the other diagonal yields an equivalent loop (bring out the mirror)?

For more fun we can try going around more than once, say

![Diagram of a loop going around twice]

or

![Diagram of a loop going around once and twice]

The first goes twice around the short way and once around the long way, and the second goes once around the short way and twice around the long way. This one

![Diagram of a loop going around three times]

goes three times around the short way and twice around the long way. This is difficult to do correctly on the donut, but is very much worth trying. When you get done, tie the loop up and count windings to make sure that there are two longs and three shorts.

You can invite speculation as to whether or not the loop is knotted (in space). The proof that it is indeed a trefoil knot is easy and fun: eat the donut and see!

Another interesting experiment of the same nature is to tie two loops on the donut corresponding to this picture.
Each goes twice around the short way and once the long way, but they are intertwined. Are the loops linked? Another donut is sacrificed to the cause of higher mathematics.

Problems and Projects:
1) Explain why you can't wrap a loop twice around the short way and no times around the long way. What about twice around both the short and long way?
2) Figure out for which number $n$ and $m$ you can wrap a loop $n$ times around the short way and $m$ times around the long way. The answer depends on divisibility properties of $m$ and $n$.
3) Show that rotation of the donut corresponds to horizontal translation of the map. What kind of transformation of the donut corresponds to vertical translation of the map?
4) How many pieces does the torus break into if you cut along a loop? The answer depends on the loop. Note that you cannot experiment on the donut because the surface attached to the inside. What is the answer on the sphere?
5) Explain why a loop that goes $n$ times around the short way and once around the long way is never knotted.
6) Experiment with other loops on the donut to see if they are knotted and/or linked. You might need some help eating all those donuts.