MATH 3040
Assignment due April 24, 2014

Projective geometry and the extended Euclidean plane

From Hilbert's treatment of the axioms of Euclidean plane, a completely worked out axiom system for geometry in the plane is quite complicated. By the nineteenth century it was realized that even the first few axioms of Euclid's treatment led to interesting results, and the problems that came up in properly drawing a picture on a flat canvas led to a somewhat different way of presenting the axiom system. In particular, the axioms of "incidence" can be stated in such a way that they are very simple and yet lead to interesting non-trivial results

By definition, parallel lines never meet. On the other hand, when one looks at or draws parallel lines, they do seem to meet. Train tracks seem to meet on the horizon. The horizon itself seems to limit the extent of the plane. So why not incorporate these feelings into the language of Geometry? Euclid's first postulate states that any two distinct points lie on a unique line. In other words they are incident to a unique line. But do two distinct lines determine a unique point? Usually, but not always. Most pairs of distinct lines intersect in a unique point. They are incident to that point of intersection. But of course, no point in sight is incident to distinct parallel lines.

Why not create new points that can be incident to parallel lines, such as the points on the horizon seem to be? In fact, we can do just that without giving up any mathematical precision, although some of the statements using this language sound a bit strange when you first hear them.

In Euclidean geometry, all lines parallel to a fixed line are parallel to each other. So the collection of all lines in the plane fall into equivalence classes, determined by the property of being parallel. (For our convenience we say that a line is parallel to itself.) In other words, we have:

(1) If L_1 and L_2 are two lines and L_1 is parallel to L_2, then L_2 is parallel to L_1.
(2) If L is a line, then we can say that L is parallel to L itself.
(3) Suppose L_1, L_2, and L_3 are lines. If L_1 is parallel to L_2, and L_2 is parallel to L_3, then L_1 is parallel to L_3.

In the language of set theory, the relation "is parallel to" is an equivalence relation, since it satisfies the symmetric, reflexive, and transitive properties, 1, 2, 3, respectively. So they determine equivalence classes.

Corresponding to each equivalence class we associate what we call a *point at infinity*, or an *ideal point*. We also say that each point at infinity is incident to every line in its corresponding equivalence class. We also say that the collection of all the points at infinity form a single *line at infinity*. Lastly, we say every point at infinity is incident to this line at infinity.

This almost corresponds to our intuition about points on the horizon. If we stand in the middle of a pair of train tracks, they appear to meet at one point one the horizon in front of us, but if we turn around, they also appear to meet at another point on the horizon. Our definition, however, only allows for one point at infinity for these parallel train tracks. In our definition, these two "points on the horizon" are identified as a single point at infinity.

Imagine a train disappearing over the horizon in one direction, only to reappear from the opposite direction.

Summarizing, we now have added infinitely many new points to the Euclidean plane, which we called the points at infinity. We will call the Euclidean points ordinary points to distinguish them from the points at infinity. We have also added one new line, the line at infinity. We call the Euclidean lines ordinary lines to distinguish them from the line at infinity.

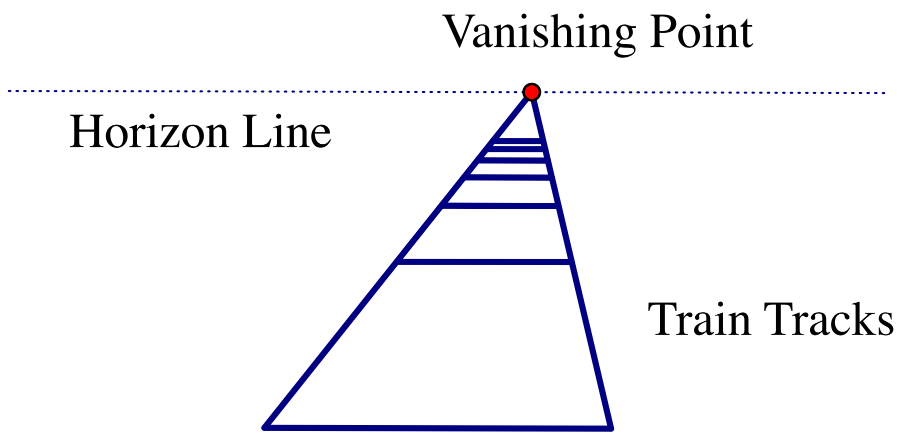

FIGURE 1

The axioms of projective geometry

After being introduced to the language of the line at infinity and its points, we realize that this "extension" of the Euclidean plane has some very simple properties. We formalize these as "axioms" again, but for a different non-Euclidean geometry which call the projective plane. So we say that a projective plane is any set that consists of two kinds of elements which are called points and lines that satisfy the following three axioms, where we have a symmetric binary relation called *incidence* between points and lines (For each point and each line, they are either incident or not incident, but not both):

Axiom 1: For every pair of distinct points there is a unique line incident to both.
Axiom 2: For every pair of distinct lines there is a unique point incident to both.
Axiom 3: There are four distinct points, where no three are incident to any line.

Axiom 3 is included to get rid of certain unwanted degenerate examples such as indicated in Figure 3.

Note that the Euclidean plane with points at infinity and the line at infinity (the extended Euclidean plane) satisfies the axioms for a projective plane. Indeed, there are many other interesting examples, we call them models, of projective planes. For instance, there are many finite projective planes, that is projective planes with a finite number of points and lines. For example, Figure 4 indicates a finite projective plane with seven points and seven lines. Of course the points do not really have to be in the Euclidean plane. We have just used the

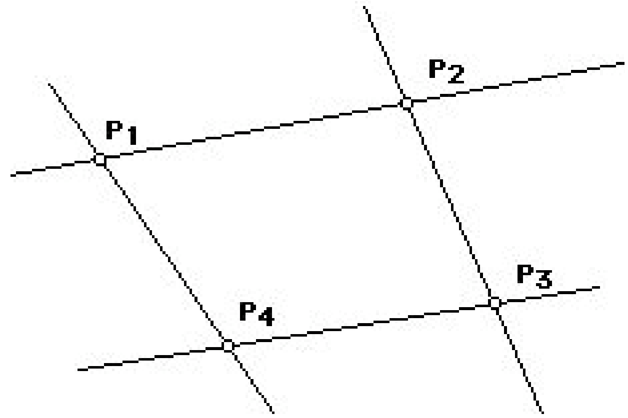

FIGURE 2. This shows Axiom 3 in the Euclidean plane (and the real projective plane).

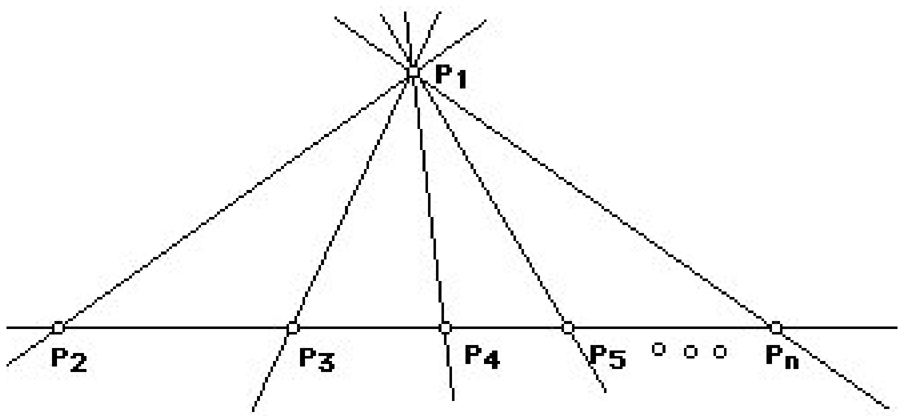

FIGURE 3. This indicates an example of a system of points that satisfy Axioms 1 and 2, but not 3.

picture to help indicate which points are incident to which lines. For example, $\mathbf{p}_1$, $\mathbf{p}_7$, and $\mathbf{p}_3$ are all the points incident to one line. The circle is meant to indicate that the points $\mathbf{p}_5$, $\mathbf{p}_6$, and $\mathbf{p}_7$ are also the points that are incident to one of the lines. All the other lines are arranged so that the incident points are on a straight line in the Euclidean plane.

1. Exercises:

In the following, assume that we have a projective plane.

1. Prove that if one interchanges the words "line" and "point" in Axiom 3, then the statement is true. Specifically, prove that there are four distinct lines, no three incident with a point.
2. Given two distinct lines, prove that there is a point that is not incident with either of them.
3. Let $\mathbf{p}$ be a point and L be a line in our projective plane, where $\mathbf{p}$ is not incident to L. Prove that there is a one-to-one correspondence between the points incident to L and the lines incident to $\mathbf{p}$.
4. Suppose that L_1 and L_2 are two lines in our projective plane and $\mathbf{p}$ is a point not incident to either line. Prove that there is a "natural" correspondence, called projection from $\mathbf{p}$,

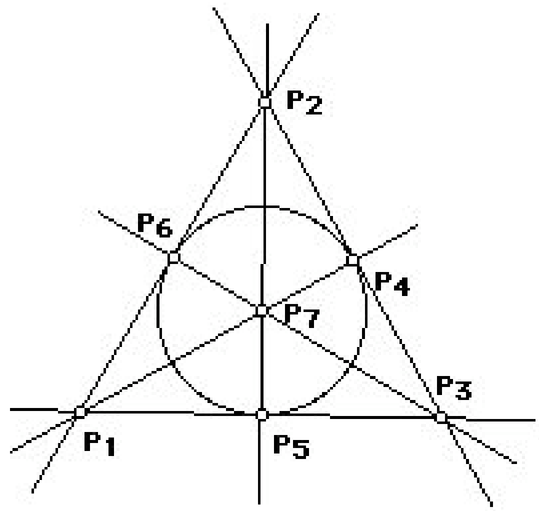

FIGURE 4. This shows the smallest possible projective plane with 7 points and 7 lines, the Fano plane.

between the points incident to L_1 and the points incident to L_2. The only thing to do here is to describe projection from $\mathbf{p}$ totally in terms of the incidence structure in our axioms of a projective plane.

5. Suppose that our projective plane has a finite number of lines. Suppose also that a point $\mathbf{p}$ has $n+1$ lines incident to it, and L is a line not incident to $\mathbf{p}$. Prove that L is incident to exactly $n+1$ points. (Hint: Use a correspondence similar to the one in Exercise 4.)
6. Suppose that our projective plane has a finite number of lines.
   (a) Prove that there is a number $n \geq 2$ such that each point is incident with $n+1$ lines, and each line is incident with $n+1$ points. (Hint: Use Exercises 4 and 5.)
   (b) Prove that the plane has n^2+n+1 points and n^2+n+1 lines. We say that n is the *order* of the finite projective plane.) Hint: Choose some arbitrary but fixed point $\mathbf{p}$. Partition the points of the projective plane into sets, where each set consists of the points on a line incident to $\mathbf{p}$, except for $\mathbf{p}$, and the set consisting of only $\mathbf{p}$. Count the number of points in each of these sets and the number of these sets using the exercises above.