

MATH 3040
Assignment for Week 10

1. Consider the set $S = \{na + mb \mid n, m \geq 0, n, m \in \mathbb{Z}\}$, for fixed $a, b \in \mathbb{N}$.
Prove that if a, b are relatively prime, then $ab - a - b \notin S$. Note that a, b are positive integers. Hint: Show that if $ab - a - b = na + mb$, then $n \equiv -1 \pmod{b}$ and $m \equiv -1 \pmod{a}$, and so $n \geq b - 1$, and $m \geq a - 1$. [This problem is due to the mathematician Frobenius, via Jeffrey Shallit.]
2. Calculate, with proof, the largest even integer not in the set S defined in Problem 1, for $a = 10, b = 14$.
3. Imagine playing a game, where Alice and Bob take turns subtracting 1, 2, 3, 4, 5, 6, 7, 8, 9 or 10 from a starting number 1000. Alice goes first and the object is to not let the number be 0 or less. With best play on both Alice's part and Bob's part, who wins, and what is Alice's choice, or does it not matter? Explain the winning strategy.
4. There are other number systems, besides \mathbb{Z} the integers, that obey all the properties (i.e. axioms) of the integers, except the well-ordering property. One such is the set of complex numbers of the form

$$\mathbb{Z}[\sqrt{-5}] = \{m + n\sqrt{-5} \mid m, n \in \mathbb{Z}\}.$$

We showed that unique factorization does not hold in $\mathbb{Z}[\sqrt{-5}]$ since $(1 + \sqrt{-5}), (1 - \sqrt{-5}), 2, 3$ are all primes in $\mathbb{Z}[\sqrt{-5}]$, while $(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3$.

Your job is to find a prime number in \mathbb{Z} that is not prime in $\mathbb{Z}[\sqrt{-5}]$. Hint: Look for primes in \mathbb{Z} of the form $(n + \sqrt{-5})(n - \sqrt{-5})$.