Math 3040, Prelim I
February 27, 2014

Please use two exam booklets, and turn them both in at the end of the Prelim. One booklet is for you final write-up of your answers and proofs. The other is for scratch work, where you can do any calculations and organizing. Only the booklet with your final answers will be graded. There are **8** questions.

1. Mathematicians in the 19th century, "knew" that a simple closed curve in the plane, with fixed length, encloses the most area when it is a perfect circle. Their "proof" was to show, correctly, that for any simple closed curve other than the circle, there was another simple closed curve, with the same length, that enclosed more area. Explain, briefly, why this does not prove that the largest area, bounded by a fixed length curve, is enclosed by a circle.

2. Use mathematical induction to show that for $n \geq 1$ the sum of the first n positive odd whole numbers is a perfect square. (Be careful to state what you are proving precisely.)

3. For x a positive real number, prove that $(\sqrt{5}+1)/2 < x$ if and only if $1+(1/x) < (\sqrt{5}+1)/2$ if and only if $x^2 - x - 1 > 0$.

4. For a real number x, the floor function $\lfloor x \rfloor$ is defined as the largest integer less than or equal to x.

   (a) Negate the statement "For all $x < y$ real, then $\lfloor x \rfloor < \lfloor y \rfloor$".

   (b) For which $x < y$ real is $\lfloor x \rfloor < \lfloor y \rfloor$?

5. The Cauchy-Schwarz inequality for real vector spaces states that for real vectors $\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u}||\mathbf{v}|$. Use the Cauchy-Schwarz inequality to prove the triangle inequality, which says that for all real vectors $\mathbf{u}, \mathbf{v}$, $|\mathbf{u} - \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$, where $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

6. Prove that there are two irrational numbers a and b such that a^b is rational. (Hint: $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is definitely rational.)

7. (a) A sequence a_n is defined by $a_1 = a$ and $a_{n+1} = 2-(1/a_n)$ for $n \geq 1$, where a is some real number. If the sequence a_n converges, determine, with proof, $\lim_{n\to\infty} a_n$.

   (b) Is the sequence a_n defined for all initial real values a? Explain.

8. Let $\mathcal{L}$ be the set of lines in the plane. For $L_1, L_2 \in \mathcal{L}$ we say $L_1 \parallel L_2$ if L_1 is parallel to L_2 or $L_1 = L_2$. Use Euclid's fifth postulate to prove that $\parallel$ is an equivalence relation. Euclid's fifth postulate says that if $L_1 \in \mathcal{L}$ and $\mathbf{p} \notin L$ is a point, then there is a unique line $L_2 \in \mathcal{L}$ such that $\mathbf{p} \in L_2$ and L_1 is parallel to L_2. We say that two lines in the plane are *parallel* if they have an empty intersection.