Math 3040
The Schroeder-Bernstein Theorem

In what follows $\mathcal{P}(X) = \{A \mid A \subset X\}$ is the set *power set* of X, the set of subsets of the set X. We say that a set function $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ is *monotone* if for all $A \subset B \subset X$, $F(A) \subset F(B)$.

Lemma (Knaster-Tarski). *If $F : \mathcal{P}(X) \to \mathcal{P}(X)$ is a monotone function, then there is an $X_0 \subset X$ that is a fixed set. Namely, $F(X_0) = X_0$.*

Proof. Define the following collection of subsets of X:

$$\mathcal{C} = \{A \subset X \mid F(A) \subset A\},$$

and

$$X_0 = \bigcap_{A \in \mathcal{C}} A.$$

Note that $\mathcal{C}$ is non-empty since $X \in \mathcal{C}$. (Thanks to Aidan for pointing out that we need to know that $\mathcal{C}$ is non-empty in order to define an intersection.) For all $A \in \mathcal{C}$, $X_0 \subset A$, and so $F(X_0) \subset F(A) \subset A$ by definition of X_0 as an intersection, that F is monotone, and the definition of $\mathcal{C}$. Since $F(X_0) \subset A$ for every $A \in \mathcal{C}$, $F(X_0) \subset X_0$, by the definition of the intersection again. Thus $X_0 \in \mathcal{C}$ by the definition of $\mathcal{C}$. Applying the monotone condition to $F(X_0)$, we get $F(F(X_0)) \subset F(X_0)$, and $F(X_0) \in \mathcal{C}$, by the definition of $\mathcal{C}$. Thus $X_0 \subset F(X_0)$, by the definition of the intersection. Thus $X_0 \subset F(X_0) \subset X_0$, and $X_0 = F(X_0)$. □

Theorem (Schroeder-Bernstein). *Let $f : X \to Y$ and $g : Y \to X$ be injective. Then there is a bijective function $h : X \to Y$.*

Proof. Define the function $F : \mathcal{P}(X) \to \mathcal{P}(X)$ by, for all $A \subset X$

$$F(A) = X - g(Y - f(A)).$$

The function F is monotone because if $A \subset B$, then

$$\begin{aligned} f(A) &\subset f(B) \\ Y - f(A) &\supset Y - f(B) \\ g(Y - f(A)) &\supset g(Y - f(B)) \\ F(A) = X - g(Y - f(A)) &\subset X - g(Y - f(B)) = F(B). \end{aligned}$$

Applying the Knaster-Tarski Lemma to F, let X_0 be a fixed set where $F(X_0) = X_0$. Then for each $x \in X$ define

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_0 \\ g^{-1}(x), & \text{if } x \in X - X_0. \end{cases}$$

Since $X_0 = F(X_0) = X - g(Y - f(X_0))$, $X - X_0 = g(Y - f(X_0))$, since X_0 and $g(Y - f(X_0))$ are complements in X. So $g^{-1}(x)$ is defined when $x \in X - X_0$ and thus h is well-defined. Define

$$h^{-1}(y) = \begin{cases} f^{-1}(y), & \text{if } y \in f(X_0) \\ g(y), & \text{if } y \in Y - f(X_0). \end{cases}$$

It is easy to check that h and h^{-1} are inverses of each other. □

Problems
Due Thursday, March 21, 2014 in class

1. Consider the two functions $f, g : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = g(n) = 2n$. By the proof of the Schroeder-Bernstein Theorem, there is a set $X_0 \subset \mathbb{N}$ such that function h above is a bijection. Find such a set X_0. That is for each $n \in \mathbb{N}$ decide whether $h(n) = 2n$ or $h(n) = n/2$, so that h is a bijection. (Hint: Write $n = 2^k m$, where m is an odd natural number and $k = 0, 1, 2, \ldots$.)

2. Prove that in the proof of the Schroeder-Bernstein Theorem the set X_0 is the smallest set such that the h defined in proof is bijective. In other words if there is another such set X_1 which works, then $X_0 \subset X_1$.

3. (Extra credit) Define a set X_1 that works for the definition of h in the proof of the Schroeder-Bernstein Theorem, replacing X_0, but such that X_1 is the largest such set. In other words if there is another such set X_2 which works, then $X_1 \supset X_2$. So h is uniquely defined, by $h(x) = f(x)$ or $h(x) = g^{-1}(x)$ if and only if $X_0 = X_1$.