Math 3040, Some Set Theory
Due March 13, 2014

Recall that a function $f : X \to Y$ is *injective* if for all $x_1, x_2 \in X, f(x_1) = f(x_2)$ implies $x_1 = x_2$, and f is surjective if for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$. A function $g : Y \to X$ is a *right inverse* for f if for all $y \in Y$, $f(g(y)) = y$, and g is a left inverse for f if for all $x \in X$, $g(f(x)) = x$.

Proposition 1. *A function $f : X \to Y$ is injective if and only if there is a function $g : Y \to X$ that is a left inverse for f and any such right inverse g is surjective.*

Proposition 2. *A function $f : X \to Y$ is surjective if and only if there is a function $g : Y \to X$ that is a right inverse for f and any such left inverse g is injective.*

Note that Proposition 2 is relatively natural, while Proposition 1 needs the axiom of choice from set theory. We say that the *cardinality* of a set X is less than or equal to the *cardinality* of a set Y and we write $\#X \leq \#Y$ if there is a function $f : X \to Y$ which is injective, or equivalently there is a function $g : Y \to X$ which is surjective. For any two sets X, Y, it turns out that $\#X \leq \#Y$ or $\#Y \leq \#X$.

We say that two sets have the same cardinality and $\#X = \#Y$ if there are functions $f : X \to Y$, $g : Y \to X$ that are both left and right inverses of each other. The following is the Shroeder-Bernstein Theorem.

Theorem 3. *For any two sets X, Y, if $\#X \leq \#Y$ and $\#Y \leq \#X$, then $\#X = \#Y$.*

Recall that for any set X, the *power set* of X is $\mathcal{P}(X) = \{A \subset X\}$, the set of subsets of X.

Theorem 4 (Cantor, Theorem 3.7 in the text)**.** *For any set X, $\#X < \#\mathcal{P}(X)$. In other words, the power set $\mathcal{P}(X)$ has a strictly larger cardinality than X.*

Proof. Clearly $\#X \leq \#\mathcal{P}(X)$, since the function $f(x) = \{x\}$ is injective. To show that the cardinalities are strict, suppose that $f : X \to \mathcal{P}(X)$ is surjective. Then define the set

$$X_0 = \{x \in X \mid x \notin f(x)\} \subset X. \tag{1}$$

It is easy to check that $X_0 \in \mathcal{P}(X)$ is not in the image of f, contradicting surjectivity of f. $\square$

Define $\#\mathbb{N} = \aleph_0, \#\mathbb{R} = \aleph_1$, and $\#\mathcal{P}(\mathbb{R}) = \aleph_2$, where $\mathbb{N} = \{1, 2, 3, \dots\}$ are the natural numbers.

Problems

1. Prove $\aleph_0 < \aleph_1 < \aleph_2$.

2. Define the function $f : \mathbb{N} \to \mathcal{P}(N)$, by $f(n) = \{n+1, 2\lfloor n/2 \rfloor\} \in \mathcal{P}(\mathbb{N})$, where $\lfloor \quad \rfloor$ is the floor function defined previously. What is the set defined by (1)?

3. Problem 3.21 in the text.

4. Find the cardinality of the set of all infinite sequences $\{(n_1, n_2, \dots, n_k, \dots) \mid n_k \in \mathbb{N}\}$ of elements of $\mathbb{N}$. As much of a proof that you can provide is appreciated.

5. Find the cardinality of the set of all finite sequences $\{(n_1, n_2, \dots, n_k) \mid n_k \in \mathbb{N}\}$ of elements of $\mathbb{N}$. As much of a proof that you can provide is appreciated.