MATH 3040
Assignment 7

1. Let $P(a), P(g)$, and $P(p)$ be the probability of the student choosing art, geology, and psychology, respectively. Then we have $P(a) + P(g) + P(p) = 1$ since the student has to choose one of them, and $P(a) = P(p) = P(g)/2$. Hence we obtain that $P(a) = P(p) = 1/4$ and $P(g) = 1/2$.

2. (a) Since $P(failure) = 1 - P(success) = 1 - 1/n$ in each experiment, the probability that there are no successes in m trials is $P(failure)^m = (1 - 1/n)^m$.

(b) Let $m = n \log 2$. Then we have

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^m = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n\log 2} = \left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n\right]^{\log 2}$$

$$= \left(e^{-1}\right)^{\log 2} = e^{\log\frac{1}{2}} = \frac{1}{2}.$$

(c) Yes. For the first bet, we have $n_1 = 6$ so that we choose m_1 to be about $6\log 2 \approx 4.158$ i.e. choose $m_1 = 4$. Similarly, for the second bet, we have $n_2 = 36$ and choose $m_2 \approx 36\log 2 \approx 24.953$ so that take $m_2 = 25$.

3. (a) No. It is $\frac{1}{4}$.

(b) $.58^2 + .11^2 + .18^2 + .13^2 = .3978$.

4. No strategy makes the probability of winning better than 1/2. In other words, before turning over any card the probability of winning is 1/2, no matter where you stop and one can show this by induction on the total number of cards. Hence, ϵ must be equal to 0 for this game to be fair.

5. It is false. When he considered the probability of drawing a black counter in a bag containing one black counter and two more counters which are either black or white, he obtained 2/3. It is equal to the probability of drawing a black counter in a bag containing two black counters and a white counter. But, it does not necessarily say that the first bag contains the same counters as the second bag. Those two probabilities have irrelevant conditions.

6. Observe that for all $n \in \mathbb{N}$, n can be uniquely represented as $2^k \cdot m$, where m is an odd number and $k = 0, 1, 2, \cdots$. Then define the set X_0 by

$$\{n \in \mathbb{N} |\ n = 2^k \cdot m,\ k = 0, 2, 4, 6 \cdots\}.$$

Now to show X_0 is the set in the proof, one needs to show either the h defined in the proof is bijective or X_0 is the fixed set where $F(X_0) = X_0$.

7. Let X_1 be another set such that the h defined in proof is bijective. Then we have $g(Y - f(X_1)) = X - X_1$ so that $F(X_1) = X_1$. Hence $X_1 \in \mathcal{C}$, as $F(X_1) \subset X_1$. Thus by the definition of X_0 we obtain $X_0 \subset X_1$.

8. Let $\mathcal{C}_1 = \{A \subset X | \, A \subset F(A)\}$ and define $X_1 = \bigcup_{A \in \mathcal{C}_1} A$. Then clearly $\mathcal{C}_1$ is non-empty since $X_0 \in \mathcal{C}_1$ so that $\mathcal{C}_1$ and X_1 are well defined. First, to show that X_1 works for the definition of h in the proof, replacing X_0, it suffices to show that X_1 is also a fixed set where $F(X_1) = X_1$. Similarly to the proof of lemma, for all $A \in \mathcal{C}_1$, $A \subset X_1$ by definition of X_1, and so we have $A \subset F(A) \subset F(X_1)$ by definition of $\mathcal{C}_1$ and the monotone condition of F. Hence $F(X_1) \supset X_1$. Also, $F(X_1) \in \mathcal{C}_1$ since $F(F(X_1)) \supset F(X_1)$ by applying the monotone condition to $F(X_1)$. Thus $X_1 \supset F(X_1)$ by definition of X_1. Therefore we conclude X_1 is a fixed set.

Now to show that X_1 is the largest such set, for every such set X_2 which works, we show that $F(X_2) = (X_2)$ in the previous problem. Hence $X_2 \in \mathcal{C}_1$ and so $X_1 \supset X_2$ by definition of X_1.

Then it is easy to check that h is uniquely defined by $h(x) = f(x)$ or $h(x) = g^{-1}(x)$ iff $X_0 = X_1$.