MATH 3040
Assignment 8

1. Let d be the diameter of a quarter. Then the edge of each square of the checkerboard is $2d$. In order to win this game, the center of the quarter must be at least its radius $d/2$ away from all the edge of the square of the checkerboard. So the probability of winning is $\frac{d^2}{(2d)^2} = \frac{1}{4}$. Now since we need to pay a quarter for each game, we expect to receive .75 dollars if we win and pay .25 dollars if we lose. So the game is fair.

2. We take the circumference as the interval $[0, 1]$ and take one point at 0 and the others at B and C in $[0, 1]$ where $B < C$. So, we have $0 < B < 1/2$ and $1/2 < C < B + 1/2$. Thus the probobility is $\frac{1/8}{1/2} = 1/4$.

3. Let $a, b \in \mathbb{N}$. First, we assume that they are relatively prime. Then we have $\gcd(a, b) = 1$. By Theorem 4.6, we obtain that there exist integers α, β such that $a \cdot \alpha + b \cdot \beta = \gcd(a, b) = 1$. Now, assume that there exist integers α, β such that $a \cdot \alpha + b \cdot \beta = 1$. Let $\gcd(a, b) = k \in \mathbb{N}$. Then k divides $a \cdot \alpha + b \cdot \beta$, which is equal to 1. Thus we obtain $k = 1$. Thus we conclude that a, b are relatively prime.

4. We will prove by induction on $n \in \mathbb{N}$. When $n = 1$, it is true by assumption. Let $k \in \mathbb{N}$ be given and suppose the statement is true for $n = k$. Then when $n = k + 1$, we apply the previous problem and have integers $\alpha, \alpha', \beta, \beta'$ such that $a \cdot \alpha + b \cdot \beta = 1$, and $a^k \cdot \alpha' + b \cdot \beta' = 1$. Then we have $a\alpha \cdot a^k \alpha' = (1 - b\beta)(1 - b\beta') = 1 - (\beta + \beta' - \beta\beta' b)b$. Therefore $a^{k+1} \cdot \alpha\alpha' + b \cdot (\beta + \beta' - \beta\beta' b) = 1$ so that a^{k+1}, b are relatively prime by the previous problem. So the statement is true for $n = k + 1$. Thus by the induction hypothesis, the statement is true for all $n \in \mathbb{N}$.

5. $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99} \equiv_5 1^{99} + 2^{99} + (-2)^{99} + (-1)^{99} + 0^{99} \equiv_5 0$. Thus the remainder is equal to 0.