MATH 3040
Assignment 1

1. Let $g(x)$ be defined by $g(x) = y_1(x) - y_2(x) = \sin(\alpha)\cos(x) + \cos(\alpha)\sin(x) - \sin(\alpha + x)$. Since α is constant, by Prop 1.7, we know $y_1(x) + y''(x) = 0$ for all x. So $y_1'(x) = -\sin(\alpha)\sin(x) + \cos(\alpha)\cos(x)$. Thus $y_1(0) = \sin(\alpha)$, $y_1'(0) = \cos(\alpha)$. $y_2'(x) = \cos(\alpha + x)$, $y_2''(x) = -\sin(\alpha + x)$. Thus $y_2(x) + y_2''(x) = 0$ for all x. $y_2(0) = \sin(\alpha), y_2'(0) = \cos(\alpha)$. Now $g''(x) = y_1''(x) - y_2''(x) = -y_1(x) + y_2(x) = -g(x)$ for all x, and $g(0) = y_1(0) - y_2(0) = 0$, $g'(0) = y_1'(0) - y_2'(0) = 0$. Thus by Lemma 1.8, we have $g(x) = 0$ for all x, and therefore $y_1(x) = y_2(x)$ for all x, which implies that $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. The proof of $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha)$ is in a similar way.

2. Let $f(x) = e^{-x}\phi(x)$. So $f'(x) = -e^{-x}\phi(x) + e^{-x}\phi'(x) = -e^{-x}\phi(x) + e^{-x}\phi'(x) = 0$. Since $f'(x) = 0$ for all x, so f is a constant function. Moreover $f(0) = e^{-0}\phi(0) = 0$, so $f(x) \equiv 0$ for all x, which implies that $\phi(x) = 0$ for all x. Thus $\phi(x) \equiv 0$.

To prove that $e^{a+b} = e^a e^b$, let $g(x) := e^{x+b} - e^x e^b$ where b is a constant. We have $g'(x) = e^{x+b} - e^x e^b = g(x)$ and $g(0) = e^b - 1 \cdot e^b = 0$. By the proposition above that we just approved, we have $g(x) \equiv 0$, which gives $e^{a+b} = e^a e^b$.

3. Let $P(n)$ be the statement for positive integer n.

Base case: when $n = 1$, $n^3 + 5n = 6$ is divisible by 3. Thus $P(1)$ is true.

Inductive step: Suppose $P(k)$ is true for some positive integer k, i.e., $k^3 + 5k = 3 \cdot a$ for some integer a. Then for $n = k + 1$, we have

$$\begin{aligned}
(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\
&= (k^3 + 5k) + 3k^2 + 3k + 6 \\
&= 3a + 3(k^2 + k + 2) \\
&= 3(a + k^2 + k + 2)
\end{aligned}$$

Thus $P(k+1)$ is true, as well. By induction, we see that for every positive integer n, $n^3 + 5n$ is divisible by 3.

3. We have $k^2 + (k^2+1) + \cdots + (k^2+k) = k^2 \cdot (k+1) + 1 + \cdots + k = k^3 + k^2 + (k+1)k/2 = \dfrac{2k^3 + 3k^2 + k}{2}$.

And $(k^2+k+1) + \cdots + (k^2+2k) = (k^2+k) \cdot k + 1 + \cdots + k = k^3 + k^2 + (k+1)k/2 = \dfrac{2k^3 + 3k^2 + k}{2}$. Thus we proved $\sum_{i=0}^{n}(k^2 + i) = \sum_{j=1}^{n}(k^2 + k + j)$ for all n.

4. Going from the left side to the right, we have

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!}\frac{k}{n+1} + \frac{(n+1)!}{k!(n-k+1)!}\frac{n-k+1}{n+1} \\
&= \frac{(n+1)!}{k!(n-k+1)!}\frac{n+1}{n+1} \\
&= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}
\end{aligned}$$

5. There is only one error thst people made in the first one. The relationship is not an equivalence relation in that the transitivity law is sometimes not satisfied. For example, we know $(1,2) \sim (0,0)$ and $(0,0) \sim (1,1)$; however $(1,3) \not\sim (1,1)$.