

MATH 3040
Assignment 4

1. Let $m = 2k + 1$, $n = 2q + 1$ for some $k, q \in \mathbb{Z}$. Then $n^2 - m^2 = (2q + 1)^2 - (2k + 1)^2 = 4(q - k)(q + k + 1)$. Since $q - k$ and $q + k$ are either odd or even at the same time, we know $2 \mid (q - k)(q + k + 1)$. Therefore $n^2 - m^2$ is divisible by 8.
2. Suppose the statement is false, i.e., suppose both a, b are odd. Let $m = 2k + 1$, $n = 2q + 1$ for some $k, q \in \mathbb{Z}$. Then $a^2 + b^2 = (2k + 1)^2 + (2q + 1)^2 = 4(q^2 + k^2 + q + k) + 2$. So $a^2 + b^2$ is not divisible by 4.

However, since c must be even as $a^2 + b^2$ is divisible by 2, c^2 is divisible by 4. A contradiction.

3. Suppose the statement is false, i.e., both $f(0) = 0$ and $g(0) = 0$. We take derivative on both sides of $x = f(x)g(x)$ and get $1 = f'(x)g(x) + f(x)g'(x)$. Let $x = 0$ and we get $1 = 0 + 0$. A contradiction.
4. We have $n^3 + 1 - (n^2 + n) = n^2(n - 1) - (n - 1) = (n - 1)(n^2 - 1) = (n - 1)^2(n + 1) > 0$ for all $n \geq 2$. Therefore $n^3 + 1 > n^2 + n$.
5. (a) By Theorem 1, there exist two positive integers $m > n$ such that $a = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$. Thus $c - a = (m - n)^2$ and $c + a = (m + n)^2$.
(b) Continuing those in (a), we have $m = \sqrt{\frac{b + c}{2}}$, $n = \sqrt{\frac{c - b}{2}}$
(c) (16, 64, 65) (33, 56, 65) (25, 60, 65) (39, 52, 65)