MATH 3040
Assignment 9

1.

(a) Let $a, b, c \in \mathbb{N}$. We define that $g \in \mathbb{N}$ is the greatest common divisor of a, b, and c and denote it by $\gcd(a, b, c)$, if both the following hold: i) g divides a, b, and c, ii) whenever $d \in \mathbb{N}$ divides a, b, and c then d also divides g.

(b) Let $g = \gcd(\gcd(a, b), c)$. We show that g satisfies the conditions i) and ii). First, since $g|\gcd(a, b)$ and $g|c$, we have $g|a$, $g|b$, and $g|c$. Second, if $d \in \mathbb{N}$ divides a, b, and c, then $d|\gcd(a, b)$ and $d|c$. Therefore, $d|\gcd(\gcd(a, b), c)$ so that $d|g$. Thus, by definition of $\gcd(a, b, c)$, g is the greatest common divisor of a, b, and c.

2.

(a) In $\mathbb{Z}_6$, we have $[0]^2 + [0]^2 = [0]$, $[0]^2 + [1]^2 = [1]$, $[1]^2 + [1]^2 = [2]$, $[0]^2 + [3]^2 = [3]$, $[0]^2 + [2]^2 = [4]$, $[1]^2 + [2]^2 = [5]$. Thus, $\{[a]_6^2 + [b]_6^2 : a, b \in \mathbb{Z}\} = \mathbb{Z}_6$. Since $1234567 \equiv 1 \pmod 6$ and there exist integers $a, b \in \mathbb{Z}_6$ such that $a^2 + b^2 \equiv 1 \pmod 6$, we can not conclude that $a^2 + b^2 = 1234567$ is impossible in this way.

(b) For $m = 8$, we have $\{[a]_8^2 + [b]_8^2 : a, b \in \mathbb{Z}\} = \{[0]_8, [1]_8, [2]_8, [4]_8, [5]_8\} \neq \mathbb{Z}_8$.

3. We have $2 \equiv 2 (\mathrm{mod}\ 10)$, $2^2 \equiv 4 (\mathrm{mod}\ 10)$, $2^3 \equiv 8 (\mathrm{mod}\ 10)$, $2^4 \equiv 6 (\mathrm{mod}\ 10)$ and $2^5 \equiv 2 (\mathrm{mod}\ 10)$. Since $1000000 \equiv 0 (\mathrm{mod}\ 4)$, we obtain $2^{1000000} \equiv 2^4 \equiv 6 (\mathrm{mod}\ 10)$.

4. Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Assume that $ab - a - b = na + mb$ for some $n, m \in \mathbb{N} \cup \{0\}$. Then we have $(a - 1 - m)b = (n + 1)a$ so that $n \equiv -1 (\mathrm{mod}\ b)$ and $m \equiv -1 (\mathrm{mod}\ a)$ because $\gcd(a, b)=1$. Hence $n + 1 = k_1 b$ and $m + 1 = k_2 a$ for some $k_1, k_2 \in \mathbb{Z}$ and $k_1 \geq 1, k_2 \geq 1$ as $n, m \geq 0$. Thus we obtain that $n + 1 \geq b$ and $m + 1 \geq a$. Now consider

$$ab - a - b = na + mb \geq a(b - 1) + b(a - 1) = 2ab - a - b,$$

which contradicts the fact that $ab > 0$. Therefore, we conclude that $ab - a - b \notin S$.

5. Let $a = 10, b = 14$. First of all, we observe that if $2k \in S$ for some $k \in \mathbb{Z}$, then $k \in \{5n+7m | n, m \geq 0, n, m \in \mathbb{Z}\}$. Then similarly we observe that if $k \notin \{5n+7m | n, m \geq 0, n, m \in \mathbb{Z}\}$, then $2k \notin S$. Since $\gcd(5, 7) = 1$, we know that $5 \cdot 7 - 5 - 7 = 23$ is not in the set $\{5n + 7m | n, m \geq 0, n, m \in \mathbb{Z}\}$, and thus $46 \notin S$. Also, we can check that any even integer greater than 46 is in the set S.

6. Alice wins with best play on both Alice's part and Bob's part. First she chooses 9, and whenever Bob chooses a number k, $1 \leq k \leq 10$, she chooses $11 - k$. Then since

$1000 - 9 = 991 \equiv 1 (\mathrm{mod} 11)$, Bob has to subtract a number between 1 and 10 from 1 at the end.

7. Consider that for a prime number 41 in $\mathbb{Z}$ we have $(6 + \sqrt{-5})(6 - \sqrt{-5}) = 41$ in $\mathbb{Z}[\sqrt{-5}]$.