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BRACING RECTANGULAR FRAMEWORKS. I

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Abstract. This paper describes the economical placing of diagonal braces in the walls and ceiling of a rectangular one story building. It begins with the definition of the structure geometry of a graph embedded in Euclidean space: a combinatorial geometry (matroid) on the set of potential braces. When the embedded graph is a plane grid of squares the geometry is graphic. Then, for example, minimal rigidifying sets of braces correspond to spanning trees in a complete bipartite graph. The methods used in the plane case are extended to analyze how sets of wall and ceiling braces in a one story building can be dependent.

1. Introduction. Interest has rekindled in the theory of the rigidity of structures. This paper uses the ideas of combinatorial geometry (matroid theory) and elementary linear algebra to study the ways to brace a one-story building. The problem began as a class project in Janos Baracs' design course in the school of architecture at the University of Montreal. Working with a model, one student analyzed all ways to brace a 3×3 one story building using four wall and four roof braces. The theory we develop below makes sense of his experimental results.

We begin by defining the general notion of a structure, an embedded graph in R^n, and its structure geometry, which describes how potential braces depend upon each other. Then we show that for a plane grid of squares the structure geometry is described combinatorially by a bipartite graph. Then we proceed to analyze how in a one story building wall and roof braces interact mechanically. That analysis makes possible the construction of taller buildings, one story at a time. In a sequel to this paper [3] one of us, Bolker, begins an analysis of the global structure of tall buildings. An earlier version of the present work can be found in [4], [5]. The exposition there is more leisurely, more suitable for architects.

2. Structures. A *structure* is a graph on a set V of vertices together with a map $p\colon V\to R^N$. The dimension N is usually 2 or 3, and we think of the points $p(V)$ as joined by rigid bars which correspond to the edges of the given graph. Assume that two points a, b move with velocities μ^a, μ^b respectively. A brace between those two points imposes a linear constraint on the vectors μ^a, μ^b: the brace permits no infinitesimal change in the distance from a to b, so

$$0=\left(\frac{d}{dt}\|(b+t\mu^b)-(a+t\mu^a)\|\right)\bigg|_{t=0}$$
$$=\left(\frac{d}{dt}\|(b-a)+t(\mu^b-\mu^a)\|\right)\bigg|_{t=0}.$$

That is true if and only if the vector $\mu^b-\mu^a$ is perpendicular to the vector $b-a$. (This is our only use of differential calculus.) Thus

$$(\mu^b-\mu^a)\cdot(b-a)=0, \tag{1}$$

or, equivalently, the two motion vectors must have equal projections on the brace:

$$\mu^a\cdot(b-a)=\mu^b\cdot(b-a). \tag{2}$$

This *mechanical principle* is the starting point for our study of structures.

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A *motion* μ of the structure is a function from V to R^N which we think of as assigning a vector μ^a at each point a so as to satisfy (2) along each brace. Since condition (2) is linear, the set M of motions is a vector space. Note that motions in M are *infinitesimal*: there are motions which are not *mechanisms*, linkages which visibly move in the prescribed manner. Figures 1(a) and 1(b) are mechanisms. The square in Figure 1(a) can be distorted by the vectors shown. The motion in Figure 1(b) is a rotation. The motion in Figure 1(c) is infinitesimal only.

The dimension of M is the number of *degrees of freedom* of the structure. M has a subspace E of dimension $N(N+1)/2$ containing the infinitesimal generators of the Euclidean motions. Figure 1(b) shows one member of E. The number of *internal degrees of freedom* is $\dim M/E = \dim M - N(N+1)/2$.

Let G be the set of edges of our original graph which are *not* part of the structure. When we wish to brace the structure we think of adding to it elements of G. For any set C of potential braces, the motion space $M(C)$ of the braced structure is a subspace of M. The codimension

$$\dim M - \dim M(C),$$

the number of degrees of freedom removed by those braces, is the *rank* $r(C)$ of that set of braces.

The rank $r(C)$ is also the rank of a certain set of vectors. It follows from the mechanical principle that each brace can be regarded as an element of the dual space M^* (more precisely, as a one dimensional subspace of M^*): only those motions are permitted which are orthogonal to the brace, so regarded. Thus any set C of cross-braces gives rise to a set of vectors in M^*, and the rank $r(C)$ is the dimension of the span of those vectors.

The set G of potential braces together with the rank function r defined above on subsets of G is the *structure geometry* (matroid) of the structure. Dependence and independence of sets of braces is vector dependence. Statements about the structure geometry can be translated into mechanical terms.

A set of braces is *independent* if the removal of any one of its members introduces a new degree of freedom. The *closure* of a set C of braces is the set of braces dependent on C: those whose addition to C removes no degree of freedom. A *circuit* is a minimal dependent set of braces. If, as is usually the case, the points of the structure lie in no proper subspace of R^N, then a set of braces *spans* if and only if the structure so braced has no internal degrees of freedom, that is, is *rigid*. A *basis* is a minimal rigidifying set of braces. A *copoint* is a maximal nonrigidifying set: one internal degree of freedom remains. We shall illustrate all these mechanical ideas in the next section, when we analyze the structure geometry for a grid of squares in the plane.

Note that any element of M^* can be thought of as a constraint on the motions of the structure. Those vectors which happen to correspond to potential braces are the ones which make up the structure geometry.

3. Grids of squares. The structure we study now is the $m \times n$ grid of unit squares in the plane. We shall allow as potential braces only the diagonals of the squares. We begin by describing the motion space. Classify the unit braces as "North–South" and "East–West", and call each sequence of points adjacent in one direction a line of points. By the mechanical principle, the points on any line can move only in such a way that their velocity vectors have equal projections on that line. The common projection is the amount by which the line moves along itself.

Thus each motion of the grid results in a directed motion of every line along itself, a scalar quantity associated with each line. Since the velocity vector at any vertex is given by the projections of that vector onto the N-S and E-W lines through that vertex, each motion of the grid is determined by the motion it produces in the lines of the grid. Furthermore, an elementary line motion, in which the points on one line move in one direction along the line, and all other points remain fixed, is a motion of the grid. We have:

THEOREM 1. *The line motions form a basis for the motion space M of the square grid.*

The dimension of M is thus the number of lines, $m+n+2$, so the grid has that many degrees of freedom. There are $m+n-1$ internal degrees of freedom; that is the rank of the structure geometry. We shall soon see which sets of $m+n-1$ braces rigidify the grid.

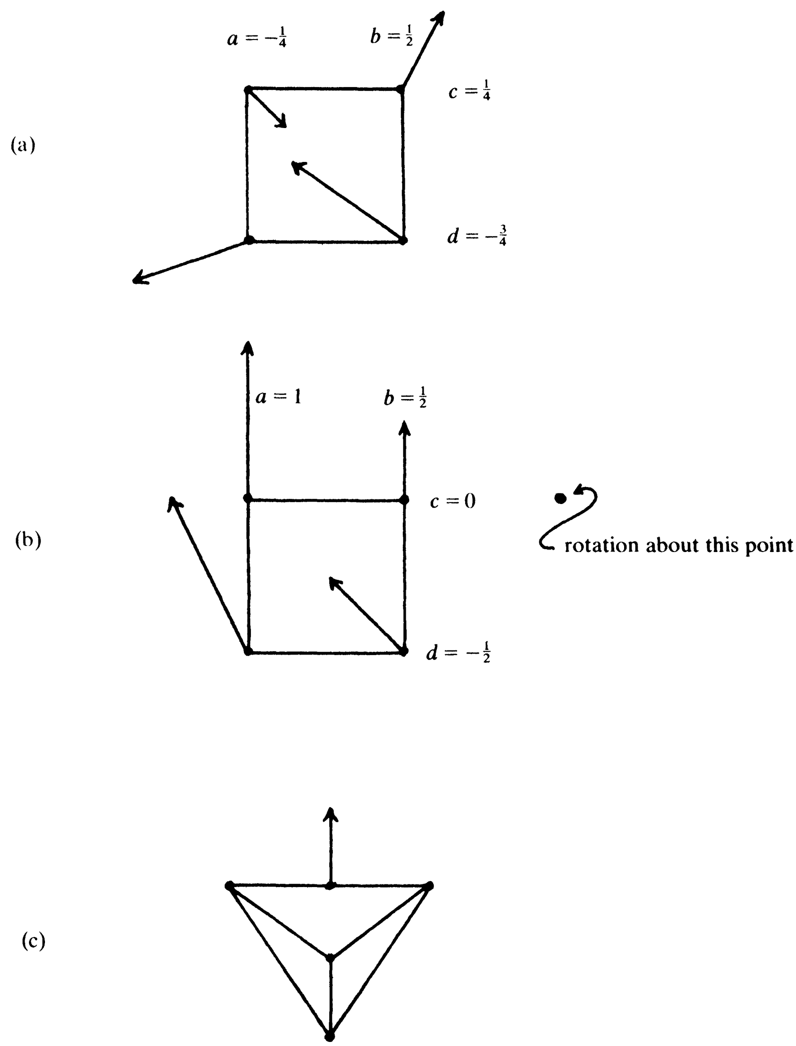


FIG. 1

Since we have coordinatized M using the line motions as a basis, we can see explicitly how the action of the dual vector corresponding to a cross-brace is represented as taking the inner product with respect to a fixed vector. Consider the square in which a cross-brace is placed. Let the motion of the N-S lines bounding the square be given by scalars a, b the motion of the E-W lines by c, d as in Figure 1(b). If the square is braced along the NE diagonal the velocity vectors at the ends of the brace are (d, a) and (c, b). These vectors have equal projections on the diagonal in the direction $(1, 1)$ so $a+d=b+c$. If we were to brace the other diagonal instead, then (c, a) and (d, b) must have equal projections on the vector $(-1, 1)$ so $-c+a=-d+b$, and $a+d=b+c$. Since the two possible cross-braces yield the same constraint $a+d-b-c=0$ on the motion space they are dependent. In the structure geometry each is in the closure of the other. We shall speak of "bracing a square", and will ignore the distinction between the two diagonals.

A motion of the grid is thus permitted when the square above is braced if and only if the inner product of that motion (written as an $m+n+2$-tuple of line motions) is orthogonal to the $m+n+2$-tuple which has a value 1 (-1) in the places corresponding to the West and South (East and North) walls of the braced square, and is 0 elsewhere.

The set of all cross braces, coordinated as above as vectors of length $m+n+2$, together form an mn by $m+n+2$ matrix of rank $m+n-1$. The case $m=n=3$ is illustrated in Figure 2.

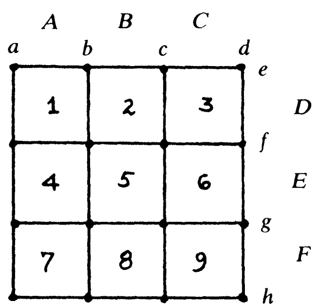


FIG. 2(a)

lines →		a	b	c	d	e	f	g	h
	1	1	−1	·	·	−1	1	·	·
	2	·	1	−1	·	−1	1	·	·
	3	·	·	1	−1	−1	1	·	·
cross	4	1	−1	·	·	·	−1	1	·
braces	5	·	1	−1	·	·	−1	1	·
	6	·	·	1	−1	·	−1	1	·
	7	1	−1	·	·	·	·	−1	1
	8	·	1	−1	·	·	·	−1	1
	9	·	·	1	−1	·	·	−1	1

FIG. 2(b)

A glance at the left and right halves of this matrix will reveal that this coordinatization of the cross-braces is the direct sum of separate coordinatizations of the N-S halls and of the E-W halls. We can better study the geometry of the rows of that matrix if we can exhibit them as vectors with two rather than four nonzero entries. The way to do that is to describe motions (modulo translations) in terms of the shears they produce in the various halls.

Let S be the real vector space of dimension $m+n$ consisting of all assignments of scalars to the halls (pairs of adjacent lines) in the $m+n$ grid of squares. We call S the space of shears. Define a linear transformation

$$\sigma: M \to S$$

on the motion space M as follows. Heading N or E down a hall, if the line to the left moves with velocity a, and that to the right with velocity b, then we say the hall undergoes a shear $b-a$. For any motion μ, make this computation for all halls. The result is a scalar assigned to each hall, an element of the space S which we call $\sigma(\mu)$.

The kernel of σ consists of those motions μ in which all the N-S lines have one velocity, all the E-W lines another. That is, the kernel is the space T of translations of the grid, a 2-dimensional subspace of the 3-dimensional subspace E of rigid motions.

If we fix one vertex q of the grid, we can compute, given any shear $s \in S$, a motion $\tau(s)$: each line moves an amount equal to the sum of the shears on the halls between it and the parallel line through q. The composite $\sigma \circ \tau$ is the identity on S, while the composite $\tau \circ \sigma$ maps each motion μ to the relative motion $\tau(\sigma(\mu))$, relative to the point q.

Since $T = \ker \sigma$ and $\operatorname{Im} \sigma \supset \operatorname{Im} \sigma \circ \tau = S$, we have a split exact sequence

$$0 \to T \to M \rightleftarrows S \to 0.$$

A rotation r with angular velocity ω has image $\sigma(r) = \omega(1, \cdots, 1)$ irrespective of the center of the rotation. Thus E is exactly the inverse image $\sigma^{-1}(\bar{\rho})$ of the one-dimensional subspace spanned by the vector $\rho = (1, \cdots, 1)$. The space M/E of internal motions is isomorphic to the quotient space $S/\bar{\rho}$.

Since each cross-brace, when regarded as a linear functional, is zero on rigid motions, and thus is zero on translations, these linear functionals are well-defined as functionals on the shear space S. The value

$$a + d - b - c = (d - c) - (b - a)$$

is the shear in the E-W hall through the braced square, less the shear in the N-S hall through that square. It is that linear functional on S which must be zero when the square is braced.

In this way, the cross-braces are coordinated in S^* as inner product with vectors with only *two* nonzero coordinates. Since the transformation σ^*: $S^* \to M^*$ is injective, the rank of any set of braces is the same computed in S^* as in M^*. For the 3×3 grid, the shear coordinatization is given in Figure 3.

halls →		A	B	C	D	E	F
	1	−1	·	·	1	·	·
	2	·	−1	·	1	·	·
	3	·	·	−1	1	·	·
	4	−1	·	·	·	1	·
cross braces	5	·	−1	·	·	1	·
	6	·	·	−1	·	1	·
	7	−1	·	·	·	·	1
	8	·	−1	·	·	·	1
	9	·	·	−1	·	·	1

FIG. 3

Observe that the set G of all cross braces, regarded as elements of S^*, spans the subspace $\bar{G} = (\bar{\rho})^{\perp}$ of codimension 1 in S^*. That is so since when all squares are braced, only rigid motions of the grid are possible.

Now we can recognize the structure geometry of the grid. Recall that for any graph, the *geometry* of the graph is that combinatorial geometry whose points are the edges of the graph and whose circuits are the polygons of the graph.

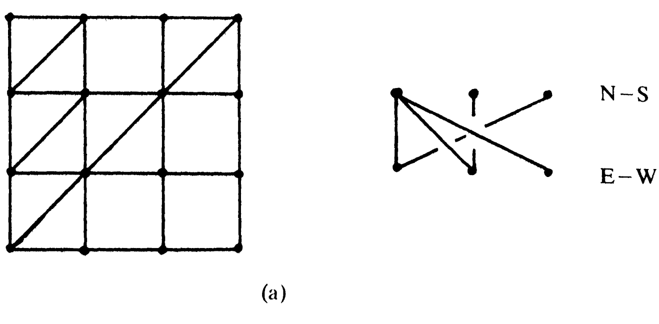
There is a standard coordinatization for the geometry of a graph which exhibits it as a vector geometry. The dimension of the coordinatizing space is the number of vertices of the graph, and each edge is coordinatized as a vector with two nonzero entries $1, -1$ in positions corresponding to its ends.

We sketch for the reader the proof that these vectors coordinatize precisely the geometry of the graph. A linear relation among such vectors (and thus among the corresponding edges) cannot involve exactly one edge at any vertex. Starting from any edge in the relation, we may proceed to an adjacent edge also in the relation, and continue until we complete a polygon all of whose edges are in the relation. So far we see that every dependent set of edges contains a polygon. But the vectors corresponding to the edges of a polygon are themselves in an obvious linear relation. Thus the minimal dependent sets in the geometry of a graph are the polygons of that graph.

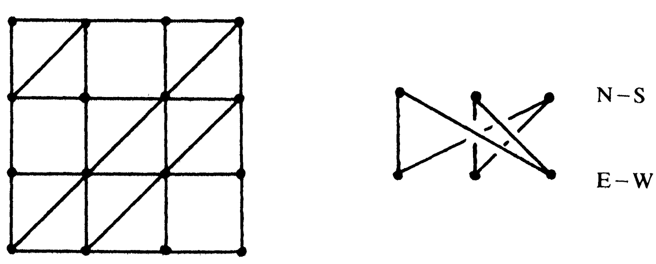
In our coordinatization for the braces of a plane grid, each brace is an element of S^* with two nonzero entries $1, -1$. This is the coordinatization of the geometry of some graph, but what graph? The vertices and edges must be the columns and rows of the matrix exemplified by Figure 3. That is, the vertices and edges of the graph are the halls and cross-braces of the grid. Since each N-S hall is related by a brace to each E-W hall, the graph in question, for the $m \times n$ grid, is the complete bipartite graph $K_{m,n}$. Since the coordinatizations agree, we have proved the following theorem.

THEOREM 2. *The geometry G of cross-braces of an $m \times n$ portion of a plane square grid is isomorphic to the geometry of the complete bipartite graph $K_{m,n}$.*

All structural information is contained in this bipartite graph. Spanning trees correspond to minimal bracing schemes (Fig. 4(a)). Circuits correspond to polygons (Fig. 4(b)). A brace b depends upon a set A of braces if and only if there is a polygon which contains the edge b in the set $A \cup \{b\}$. The closure $\bar{A}$ of A is obtained by adjoining to A all braces which depend on it. $\bar{A}$ can be constructed inductively by adding at each stage those edges which, together with edges already in or added to A, complete a quadrilateral.



(a)

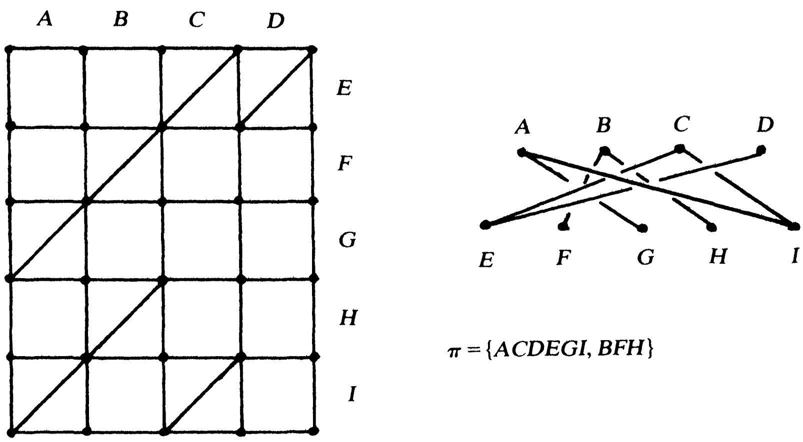


(b)

FIG. 4

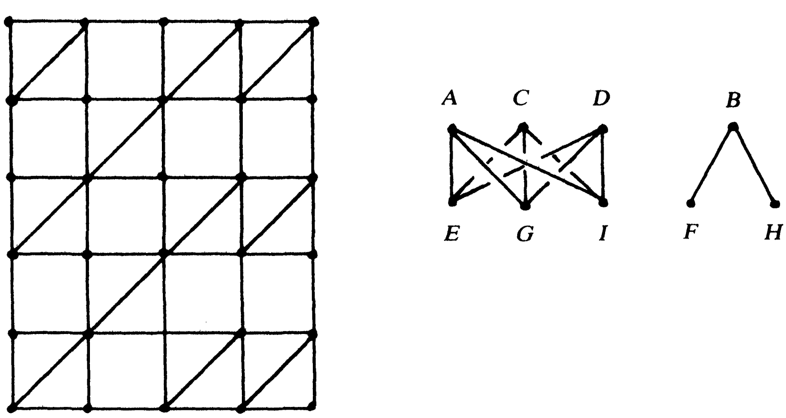
All symmetries of the complete bipartite graph are symmetries of the geometry of cross-braces of the grid. The rank of a set of cross-braces is completely unaffected by an arbitrary permutation of the E-W or of the N-S halls of the grid. Even in a fairly large grid, it is easy to list, up to these symmetries, all the combinatorially distinct circuits or minimal bracing schemes. Moreover, the symmetry proves the odd fact that the squares on the perimeter of the grid have no special structural significance by virtue of their special position.

Next we shall study the rank of a set A of braces. When A is regarded as a subgraph of $K_{m,n}$ it determines a partition $\pi = \pi(A)$ of the halls (vertices) into edge connected components. It is easy to compute π directly from the grid, without first drawing the corresponding graph (Fig. 5(a)). Sets A and A' determine the same partition if and only if they have the same closure, so partitions into edge connected components (some of which may contain isolated vertices) correspond to closed sets of braces. Let $|\pi|$ be the number of parts of π.



An independent set of rank 7

(a)



and its closure, a copoint

(b)

FIG. 5

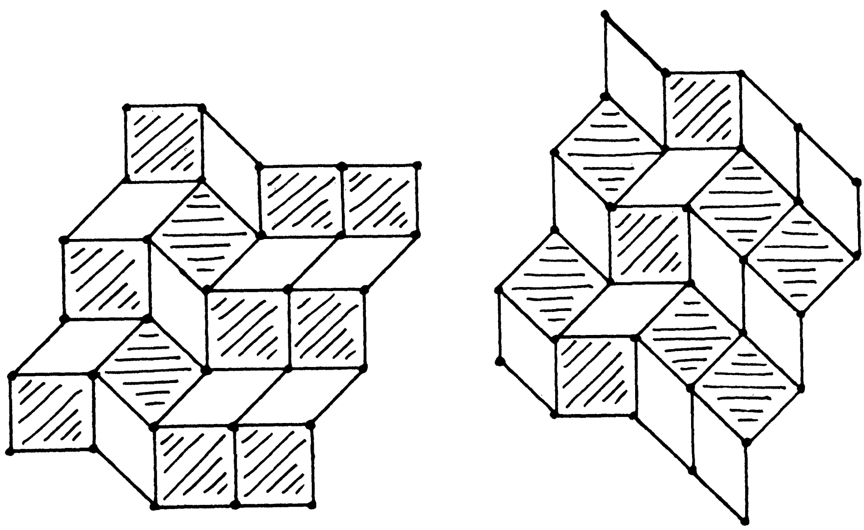
LEMMA 3. *Let $\pi = \pi(A)$. The space $S^\pi \subset S$ of shears permitted when A is braced is the space of vectors of shears constant on the parts of π.*

$$\text{(3)} \qquad \operatorname{Dim} S^\pi = |\pi| \quad \textit{and} \quad r(A) = r(\pi) = m + n - |\pi|.$$

Proof. We have seen that each brace in A constrains motions by forcing the shears in the two halls containing the brace to be equal. Since equality is transitive, S^π is as claimed; its dimension is clearly $|\pi|$. The assertion about $r(A)$ follows since $r(A)$ is the codimension of $M(A) = S^\pi \subset S$. □

Equation (3) is really a statement about the homology of the subgraph A of $K_{m,n}$, and so follows directly from Theorem 2. We have chosen to prove it as a part of Lemma 3 to keep its mechanical significance nearer the surface.

We can take further advantage of Lemma 3 and the fact that every motion of a grid of squares integrates to a mechanism to picture vividly the motions in S^π. The squares corresponding to a particular part of π must undergo the same rotation in any such motion (Fig. 5(c)).



$\pi = \{ACDEGI, BFH\}$

(c)

$\pi = \{ACDFH, BEGI\}$

(d)

FIG. 5 (*cont.*)

A copoint (Figure 5) of the structure geometry has rank $m+n-2$ and $|\pi| = 2$ (Fig. 5(b)). It leaves one internal degree of freedom. The complementary set of squares, a *bond*, are deformed into congruent rhombi, in two different orientations corresponding to the two components of the graph complementary to the copoint (Fig. 5(c)). Moreover in a bipartite graph the complement of a copoint π is again a copoint provided each of the two parts of π contains an edge. Thus we can draw Figure 5(d) dual to Figure 5(c). Figure 6 is self dual in this sense.

Consider a set of squares at the corners of a rectangle. They form a rank 3 subgeometry of the structure geometry: all four braces form a circuit while any three are a basis. Two diagonally opposite braces are a copoint in the rank 3 geometry of those corner braces, so the other two squares will always be congruent rhombi, rotated relative to each other, whatever motions are applied to the whole grid. A copoint of this kind transmits information from one corner of the grid to the other. (Fig. 7.)

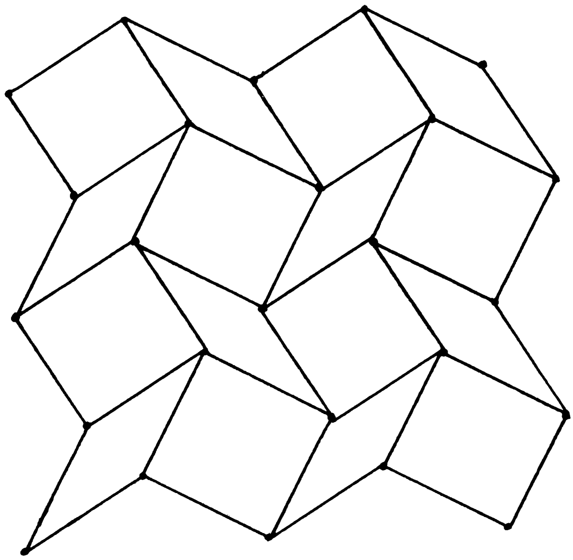
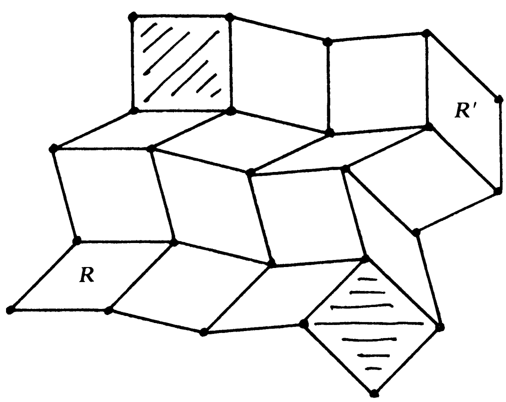


FIG. 6



Rhombi R and R' are congruent.

FIG. 7

Finally, let us define a *simple* motion of the grid as one which deforms a minimal set of squares, and thus leaves square a copoint of squares. Note that every motion of the grid leaves square some closed set of squares, but not necessarily a copoint of squares. A motion given by the shear down a single hall is simple, so simple motions span the motion space, but most motions are not simple. For instance, a line motion is simple only if the line is on the perimeter of the grid.

4. The one-story building. A one-story building is an $m \times n$ grid of squares supported by vertical bars of equal length over *fixed* points in a plane. Let μ be a motion of the building. Since μ is 0 at the base of each vertical member, the mechanical principle says μ must be horizontal at the top. Thus μ is a motion of the roof in its own plane. The motion space M for the building is isomorphic to the motion space for the roof grid. However, since the entire floor is fixed, all motions of the roof are internal motions of the building. There are thus $m+n+2$ internal degrees of freedom.

To brace the building, we place cross-braces in certain wall and roof panels. A wall brace *prevents the line of that wall in the roof from moving along itself.* The line is *blocked*: the line motion is 0. Note that two wall braces anywhere along the same wall form a dependent set. Any brace in a wall braces the whole wall. Since the line motions are independent, any set of wall braces in different walls is independent.

Next we consider the effect on shears of blocking lines. Blocking two perpendicular lines has no effect on shears, but prevents all translations of the ceiling relative to the floor. We can then use S, the space of shears, for the motion space. Architecture demands, and we shall assume, that two such lines be always blocked, so that we can analyze the structure geometry of the building as a vector geometry in S^*. Observe that when two perpendicular lines are blocked their intersection in the roof is pinned. It cannot move.

Let B be a set of wall braces. Let $\tau=\tau(B)$ be the partition of the set of halls between braced walls into regions between consecutive braced walls. Halls which are outside all braced walls do not appear in $\bigcup \tau$. Since we are assuming B contains two perpendicular walls, $|\tau|=|B|-2$.

LEMMA 4. *Let $\tau=\tau(B)$. The space $S_\tau \subset S$ of shears permitted when B is braced is the space of vectors of shears summing to* 0 *on the parts of τ.*

$$r(B)=r(\tau)=|\tau|=m+n-\dim S_\tau.$$

Proof. The relative motion of two parallel lines is the sum of the shears between them. Thus if two such lines are blocked, the shears between them must sum to 0. Since, as we have observed, a set of blocked lines is an independent set of constraints on motions, the space of motions allowed when B is braced has codimension $|B|$ in M, and hence codimension $|B|-2=|\tau|$ in S. □

Let A be a set of roof braces and B a set of wall braces. Recall how in § 3 we constructed the partition $\pi(A)$ of the set of halls. The motions permitted when $A\cup B$ is braced depend only on $\pi(A)$ and $\tau(B)$:

$$M(A\cup B)=M(\pi,\tau)=S^\pi\cap S_\tau,$$

and

$$r(A\cup B)=r(\pi,\tau)=m+n-\dim(S^\pi\cap S_\tau).$$

We shall study that rank by seeing how the roof braces A cause dependencies in the wall braces B. That is, we shall study

$$r^\pi(\tau)=\dim S^\pi-\dim(S^\pi\cap S_\tau). \tag{4}$$

Then the full rank can be recovered using

$$r(\pi,\tau)=r(\pi)+r^\pi(\tau).$$

Figure 8 is a lattice diagram of the relevant subspaces of S.

THEOREM 5.

$$r^\pi(\tau)=\dim(\textit{Actions of } S_\tau^\perp \textit{ on } S^\pi).$$

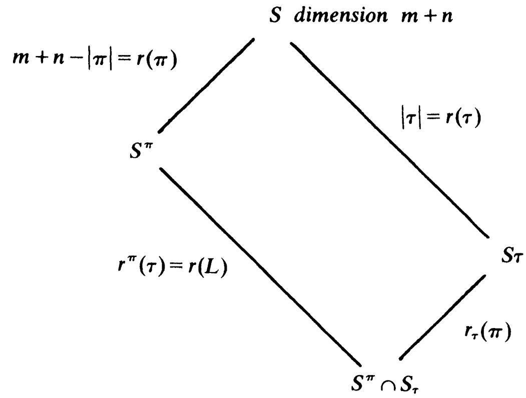


FIG. 8

Proof. By "Actions of $S_\tau^\perp$ on S^π" we mean the vector space of linear functionals on S^π obtained by restricting the functionals in $S_\tau^\perp \subset S^*$ to S^π. One way to prove the theorem is to make the routine argument in linear algebra which produces the natural isomorphism

$$S^\pi / S^\pi \cap S_\tau \cong \text{Actions of } S_\tau^\perp \text{ on } S^\pi.$$

A second way, more appropriate here, is to use a mechanical interpretation. $S_\tau^\perp \subset S^*$ is the space spanned by the vectors corresponding to braced walls. When the roof braces in the closed set corresponding to π are in place the motion space for the grid is S^π rather than S. Dependence among the vectors of $S_\tau^\perp$ thought of as constraints on motion is their dependence as functionals on S^π. □

DEFINITION 6. *The joint occupancy matrix $L = L(\pi, \tau)$* of the two partitions π, τ is defined by

L_{ij} = number of halls (vertices) in the intersection of the ith part of π with the jth part of τ.

THEOREM 7. $r^\pi(\tau) = r(L)$.

Proof. For any subspaces $W \subset S$, $V \subset S^*$ the rank of V as a space of actions on W is the rank of the matrix $f^j(x_i)$ where $\cdots f^j \cdots$ is a basis for V and $\cdots x_i \cdots$ a basis for W. If we choose for S^* the basis dual to the standard one for S, so that functionals act by taking inner products, we see that the characteristic vectors of the parts of τ form a basis for $S_\tau^\perp$ (Lemma 4), while the characteristic vectors of the parts of π form a basis for S^π (Lemma 3). But the inner product of two such vectors, one from each basis, is the cardinality of the intersection of the corresponding parts of π and τ. □

In the original version of this paper [3] we concentrated on $r_\tau(\pi) = \dim S_\tau - \dim (S^\pi \cap S_\tau)$, the rank of a set of roof braces modulo a set of wall braces. The difference $r(\pi) - r_\tau(\pi)$ is the reduction in rank of a set of roof braces caused by the bracing of certain walls.

THEOREM 8.

$$r(\pi) - r_\tau(\pi) = |\tau| - r(L),$$

the column nullity of L.

Proof. For the proof see Fig. 8.

THEOREM 9. $\operatorname{Dim} S^\pi \cap S_\tau = |\pi| - r(L)$, *the row nullity of L. The orthogonal complement of the column space of L is naturally isomorphic to $S^\pi \cap S_\tau$.*

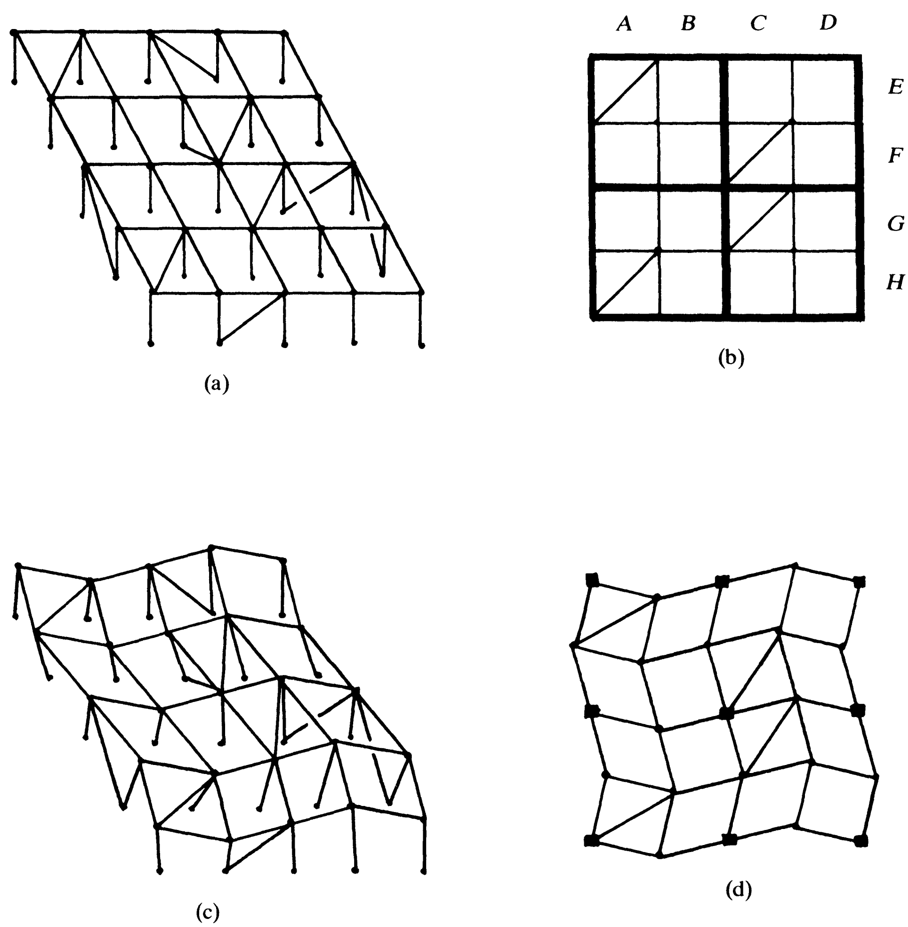
Proof. The first statement follows from Theorem 7, Equation (4) and Lemma 3. The second is easy to prove. Suppose $s \in S^{\pi}$. Then s is constant on each part of π. Say it has value s_i' on the ith part. Then the inner product of the vector s' with the jth column of L is the inner product of s with characteristic vector of the jth part of τ. Now $s \in S^{\pi} \cap S_{\tau}$ if and only if the shears in s sum to 0 on every part of τ, if and only if s' is perpendicular to every column of L. □

COROLLARY 10. *An independent set A of roof braces and a set B of wall braces together minimally brace a one-story building if and only if $L(\pi(A), \tau(B))$ is nonsingular.*

Proof. The structure is rigid if and only if $S^{\pi} \cap S_{\tau} = \{0\}$, or, equivalently, the row nullity of L is 0. The independent roof braces remain independent modulo the wall braces if and only if the column nullity is 0 (Theorem 8). □

To see how these ideas work in practice, consider the building shown in perspective in Fig. 9(a) and schematically in Fig. 9(b), where braced walls are indicated with heavy blocked lines. Then

$$\pi = \{AEH, B, CFG, D\} \quad \text{and} \quad \tau = \{AB, CD, EF, GH\}.$$



(a) (b)

(c) (d)

FIG. 9

The joint occupancy matrix L is

$$\begin{array}{c|cccc|} & \begin{matrix}A\\B\end{matrix} & \begin{matrix}C\\D\end{matrix} & \begin{matrix}E\\F\end{matrix} & \begin{matrix}G\\H\end{matrix} \\ \hline AEH & 1 & 0 & 1 & 1 \\ B & 1 & 0 & 0 & 0 \\ CFG & 0 & 1 & 1 & 1 \\ D & 0 & 1 & 0 & 0 \\ \hline \end{array};$$

it has rank 3. The roof braces by themselves are independent, but become dependent with nullity $1 =$ column nullity of L modulo the wall braces. The vector $(-1, 1, 1, -1)$ is orthogonal to every column of L. The corresponding shear, drawn in perspective in Fig. 9(c) and as a deformation of the roof in Fig. 9(d) takes the value -1 in halls A, D, E, H and the value 1 in halls B, C, F, G. This is an infinitesimal motion only: nine nodes in the roof are pinned in space by the braced walls.

Were the roof brace in hall G moved from hall C to hall D, the joint occupancy matrix would be

$$\begin{array}{c|cccc|} & \begin{matrix}A\\B\end{matrix} & \begin{matrix}C\\D\end{matrix} & \begin{matrix}E\\F\end{matrix} & \begin{matrix}G\\H\end{matrix} \\ \hline AEH & 1 & 0 & 1 & 1 \\ B & 1 & 0 & 0 & 0 \\ CF & 0 & 1 & 1 & 0 \\ DG & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

which is nonsingular. The 10 braces, 6 in the walls and 4 in the roof, minimally brace the building.

When a fixed pattern of braced walls is used, addition of roof braces one at a time either leaves the partition π unchanged, in which case the new brace depends on the roof braces already in place, or it joins two parts of π and alters L by replacing two rows by their sum. Thus starting with the discrete partition we can produce minimal bracings by adding roof braces one at a time, keeping L column independent at each step. Figure 10 shows three examples of bracing schemes which might be used on three successive floors of a 5×5 building in which the outer two walls were braced on all sides on each floor.

We know the circuits in the geometry of roof braces: they are the polygons in $K_{m,n}$. There are no nontrivial circuits in the geometry of wall braces: any set is independent. The joint occupancy matrix allows us to find the circuits in the full structure geometry of the building. Suppose A is a circuit in the geometry of the roof braces modulo a given set B of wall braces. Then for some $B' \subset B$, $A \cup B'$ is a circuit in the full structure geometry. How can we determine B'? Consider removing a brace b from B. If b is in an extreme braced wall, we eliminate a column of L. If b is internal, we merge two parts of τ and replace two columns of L by their sum. The brace b is in the circuit with A if and only if its removal does not change the rank of $A \cup B$ but does change the rank of A modulo B. Thus removing such a brace reduces the column nullity of L by 1, and reduces the number of columns by 1, so it preserves the rank of L.

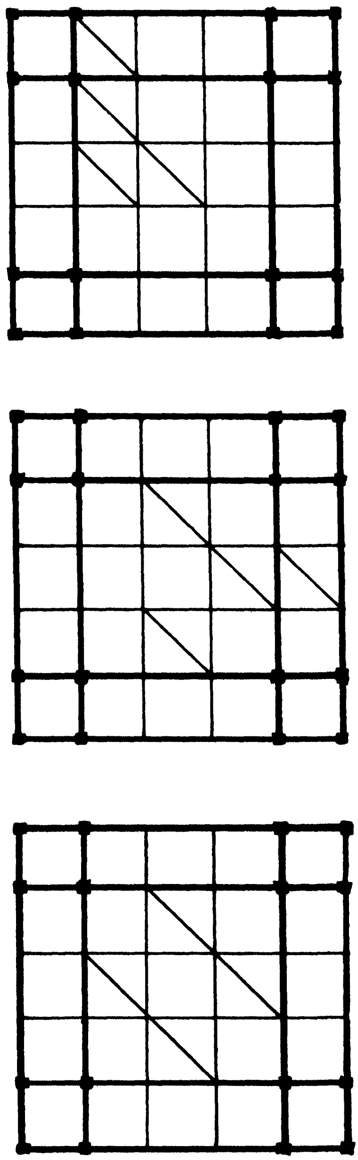


FIG. 10

For example, in Figure 11, the roof braces form a circuit modulo the given wall braces, and the joint occupancy matrix

$$\begin{array}{r|cccc|}
 & \begin{matrix}A\\B\end{matrix} & \begin{matrix}C\\D\end{matrix} & \begin{matrix}E\\F\end{matrix} & \begin{matrix}G\\H\end{matrix} \\
\hline
ABDEFH & 2 & 1 & 2 & 1 \\
C & 0 & 1 & 0 & 0 \\
G & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

has rank 2, column nullity 1. Removal of wall brace 3 eliminates the column GH, which

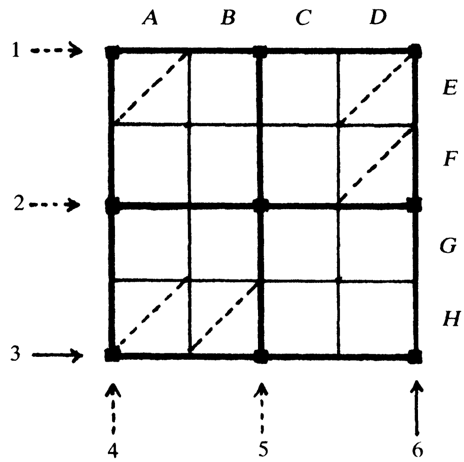


FIG. 11

reduces the rank of the matrix to 2 and leaves the column nullity unchanged. Thus wall brace 3 (and similarly wall brace 6) are *not* in the circuit. Removal of wall brace 2 merges the parts EF and GH of τ and produces a nonsingular matrix, with rank unchanged. Thus wall brace 2 (and similarly wall braces 1, 4 and 5) is in the circuit. Note that it is a circuit formed by exchange at the brace BF between the circuit AE, BF, 1, 2, 4, 5 and the circuit formed in the roof by adding the brace BF.

In our earlier paper [3] we studied the dependencies among the braces of a roof tree T (minimal rigidifying set for the roof) caused by the bracing of certain walls. Let us derive some of the results from that paper using our new methods.

Suppose we brace the four outer walls of a building. Then $\tau = \{\{\text{E-W halls}\}, \{\text{N-S halls}\}\}$ and $\pi = \{\text{all halls}\}$. The joint occupancy matrix is $\boxed{m, n}$, which has column nullity 1. Thus there is a unique circuit C using all four wall braces and some of the braces $t \in T$. How can we decide which? Let e be an edge in T. When we remove e the partition $\pi = \{\text{all halls}\}$ splits into two parts. Let m' and n' be the number of E-W halls and N-S halls in one part.

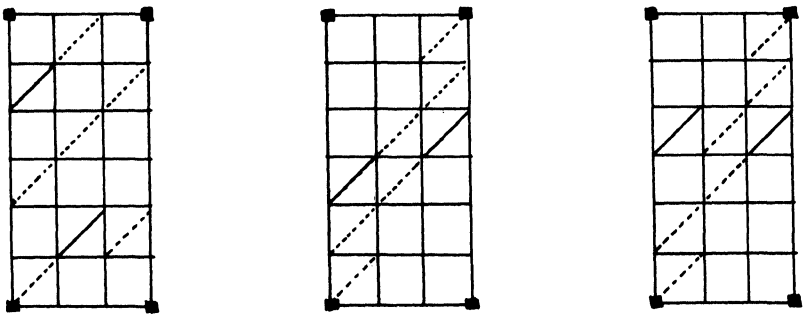
THEOREM 11. *The roof braces in the circuit C are those for which the vector (m', n') is not a multiple of (m, n).*

Proof. When e is removed the joint occupancy matrix becomes

$$L' = \begin{array}{|cc|} \hline m' & n' \\ m-n' & n-n' \\ \hline \end{array}.$$

The circuit C contains e if and only if the removal of e does not change the rank of T, or, equivalently, the column nullity of L' is one less than that of L. Since L has column nullity 1, e is in C if and only if L' is nonsingular. The theorem then follows. □

Theorem 11 has intersting consequences. First, permutation of rows and columns leaves all structural information invariant. Thus in Figures 12(a) and 12(b) the dotted lines indicate the roof braces which, together with the outside braced walls, form a circuit. In fact, the structure Fig. 12(c) has a structure geometry isomorphic to that for the structures in Figs. 12(a) and 12(b) even though it is not the result of permuting halls, because the distribution of E-W and N-S halls in the parts of the partitions obtained by deleting edges one at a time in turn is the same in both cases.



(a) (b) (c)

FIG. 12

Second, Theorem 11 conveys the most information when the greatest common divisor of m and n is large. For a square grid it shows that the set of braces on the diagonal together with the four wall braces is a circuit. In fact, similar circuits occur whenever the diagonal of the building meets intersections of walls.

When m and n are relatively prime any roof tree forms a circuit with the four braced outside walls, since (m', n') can never be a multiple of (m, n). An architect can minimally brace such a building by removing any brace from any roof tree. If she leaves a full roof tree she has a *spanning circuit*, which is architecturally useful because the building remains rigid if any single brace fails.

5. Buildings with different shapes. The theory in § 4 covers one story buildings with nonrectangular floor plans as long as they are *wall-convex*: every wall is connected. Such a building is shown in Fig. 13. The roof tree illustrated has nullity 1 and contains a

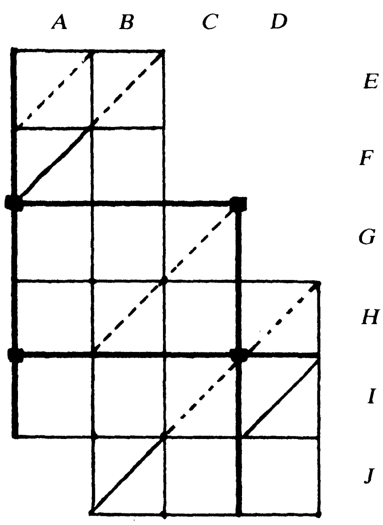


FIG. 13

unique circuit. Since $\tau = \{GH, ABC\}$ the analysis following Theorem 11 applies with $m = 2$, $n = 3$, even though it is not the outside walls which are braced. Breaking the tree at each brace in turn produces only two vectors (m', n') which are multiples of $(2, 3)$: AF and BJ both yield $(0, 0)$. Thus the circuit consists of all the other roof braces, shown dotted, and the four wall braces. Buildings with courtyards and parallel wings are not wall convex. We leave to the reader the analysis of bracing schemes for the building in Fig. 14.

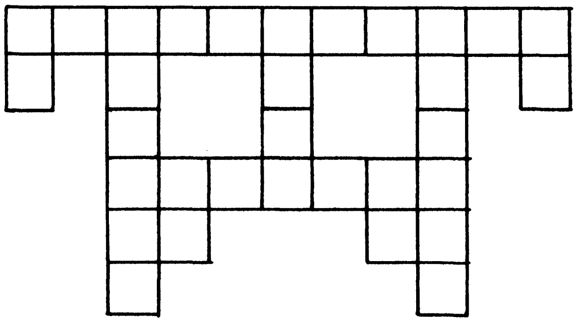


FIG. 14

Let us return now to a rectangular $m \times n$ floor plan, but allow the rooms themselves to be rectangular rather than just square. Suppose the ith hall has width h_i, $i = 1, \cdots, m+n$. The argument in § 3 which shows that the line motions form a basis for the motion space M is unchanged, as is the definition of the map $\sigma: M \to S$, onto the space of shears.

What is the vector in S^* whose orthogonal complement in S contains the motions permitted when the rectangle in Figure 15 is braced by its N-E diagonal? In that case,

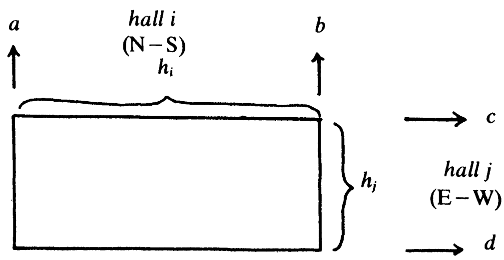


FIG. 15

the vectors (d, a) and (c, b) must have equal projections on the diagonal which has direction (h_i, h_j), so the vector $(d-c, a-b)$ must be perpendicular to (h_i, h_j). Thus $h_i \cdot (d-c) + h_j \cdot (a-b) = 0$, or, equivalently (and more useful later)

$$\frac{d-c}{h_j} - \frac{b-a}{h_i} = 0.$$

That suggests that we change our basis in S, replacing the unit shear s_i in hall i by the shear $h_i s_i$. Since $b-a$ and $d-c$ are the coefficients of s_i and s_j, $(b-a)/h_i$ and $(d-c)/h_j$ are the coefficients of our new basis vectors. Thus, relative to our new basis for S and its dual basis, the brace in the rectangle above corresponds to the vector in S^* with two nonzero entries, -1, and 1, in the place corresponding to the two halls. Thus the structure geometry for this grid is again the graphic geometry $K_{m,n}$. Theorem 2 remains true.

It is no surprise that the dimensions of the rooms are irrelevant in the plane grid. They do matter when we brace a one-story building on that grid. To see that we must investigate how the wall braces act as constraints on shears. Lemma 4 must be modified.

Relative to our new basis in S, a vector x of shears is permitted when B is braced if and only if for each part τ_j of τ,

$$\sum_{h_k \in \tau_j} h_k x_k = 0. \tag{5}$$

Theorem 5 remains true, but Definition 6 must be changed.

DEFINITION 12. The *joint occupancy matrix* $L = L(\pi, \tau)$ of the two partitions π, τ is defined by

L_{ij} = the sum of the widths of the halls in the intersection of the ith part of π with the jth part of τ.

THEOREM 13. *Theorems* 7–10 *remain true for buildings with rectangular rooms when the new joint occupancy matrix is used.*

Proof. Use in S the basis introduced above. Then relative to the dual basis in S^*, the functional expressing the condition (5) is given by

$$(f^j)_k = \begin{cases} h_k & \text{if the } k\text{th hall is in } \tau_j, \\ 0 & \text{otherwise.} \end{cases}$$

That, together with the altered Lemma 4, is enough to make the rest of the proofs of Theorems 7–10 go through, mutatis mutandis. □

REFERENCES

[1] L. ASIMOW AND B. ROTH, *The rigidity of graphs*, Trans. Amer. Math. Soc., to appear.
[2] J. BARACS, *Rigidity of articulated panel structures*, Bull. Int. Assoc. for Shell and Spacial Structures 59 Vol XVI-3, Madrid, 1975.
[3] E. BOLKER, *Transportation polytopes*, J. Combinatorial Theory, 13 (1972), pp. 251–262.
[4] ———, *Bracing grids of cubes*, Environment and Planning B, 4 (1977), pp. 157–172.
[5] E. BOLKER AND H. CRAPO, *How to brace a one-story building*, Environment and Planning B, 4 (1977), pp. 125–152.
[6] H. CRAPO, *More on the bracing of one-story buildings*, Ibid., 4 (1977), pp. 153–156.
[7] H. CRAPO AND G.-C. ROTA, *Combinatorial Geometries*, Preliminary Edition, MIT Press, Cambridge, MA 1970.
[8] B. GRÜNBAUM, *Lectures in lost mathematics*, Lecture notes, Univ. of Washington, Seattle, 1975.
[9] W. WHITELEY, *Introduction to structural geometry I–IV*, Preprint, Champlain Regional College, St. Lambert, Québec, 1977.