Notes and books are allowed; any collaboration is not allowed.

1. (10) Find the image of the unit disc under the action of $e^A$, $A = \frac{7}{30} \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Hint: $e^{0.7} \approx 2$.

2. (10) Plot the orbits of a system $\dot{x} = Ax$ taking the eigenbasis of $A$ as an orthonormal one;

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \\ 3 & 4 & 0 \end{pmatrix}.$$

3. (6) Solve the Cauchy problem for the heat equation on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$: $u_t = u_{xx}$, $u|_{t=0} = \sin^4 x$.

4. (10) Find the solutions and plot the orbits in the $x$-plane for the following Cauchy problem: $\ddot{x} + Ax = 0$, $x(0) = a$, $\dot{x}(0) = b$

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \quad a = e_1, \quad b = 0$$

5. (14) Is an operator

$$\Phi : f \mapsto g, \quad g(t) = t - \frac{t^4}{4} + \int_0^t \tau^2 f(\tau) d\tau$$

contracting in $C[-\frac{1}{2}, \frac{1}{2}]$? Find all its fixed points.

Hint: find several Picard approximations with a well chosen first one, and guess the form of the function that is the fixed point of the operator.