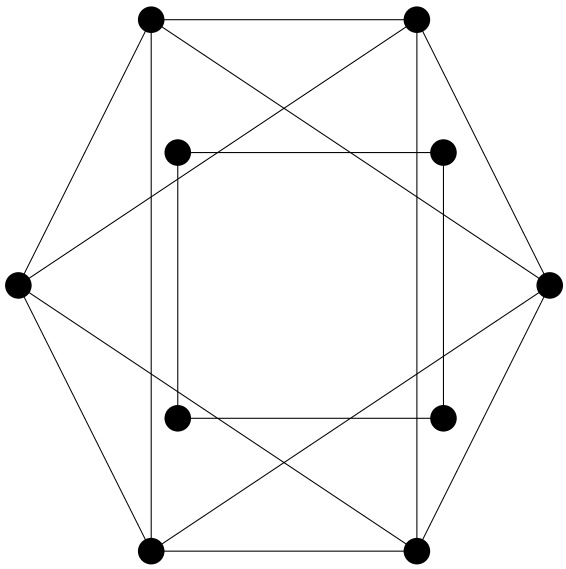
1. Is this graph Eulerian?



No, the graph is not Eulerian, because it is not connected. None of the four vertices in the center that form a square are adjacent to any of the six outer vertices.

2. Give the best upper bound for the Ramsey number $N(3,3,3,3)$ that you can.

We know that $N(3,3) = 6$, as was presented in class. Therefore, $N(3,3,2) = N(3,2,3) = N(2,3,3) = 6$, to have no 2-clique of a given color means that that color cannot be used to color any edges at all. By Ramsey's theorem,

$$N(3,3,3) \leq N(3,3,2) + N(3,2,3) + N(2,3,3) + 2 - 3 = 17.$$

We can repeat this process to get

$$N(3,3,3,2) = N(3,3,2,3) = N(3,2,3,3) = N(2,3,3,3) = N(3,3,3) \leq 17.$$

If we apply Ramsey's theorem again, we get

$$N(3,3,3,3) \leq N(3,3,3,2)+N(3,3,2,3)+N(3,2,3,3)+N(2,3,3,3)+2-4 \leq 66.$$

The exact value of $N(3,3,3,3)$ is not known. The best bounds I could find online were $51 \leq N(3,3,3,3) \leq 62$. Proving an upper bound of 66 was sufficient for full credit.

3. How many labeled trees are there with three vertices that are not leaves: one of degree 2, one of degree 3, and one of degree 4?

Suppose that the tree has n leaves. Each leaf has degree one, so the degree sum is $2+3+4+n = 9+n$. Since the tree has $n+3$ vertices, it has $n+2$ edges, and hence degree sum $2n+4$. Therefore, $2n+4 = 9+n$, from which $n = 5$.

Since there are five leaves, the tree has eight vertices. There are eight ways to pick the vertex of degree 2, then 7 ways to pick the vertex of degree 3 (as it cannot be the same as the vertex of degree 2), and then 6 ways to pick the vertex of degree 4, for 336 ways to pick which three vertices are not leaves.

Each labeled tree corresponds to a unique Prufer code, which is a way to list the vertex of degree 2 once, the vertex of degree 3 twice, and the vertex of degree 4 three times in six spots. There are $\binom{6}{3} = 20$ ways to pick which three positions in the Prufer code have the vertex of degree 4. Once this is chosen, there are 3 ways to pick the position of the vertex of degree 2, and then the vertex of degree 3 must go in the other two positions. Thus, for a given way to pick the degrees of the vertices, there are 60 labeled trees.

There are 60 labeled trees for each way to pick the degrees of the vertices, so there are $(336)(60) = 20160$ labeled trees with the specified degrees.

4. A graph has 10 vertices and 20 edges. Determine its largest possible girth.

By Theorem 4.2, since the graph has 10 vertices than $\frac{1}{2}10\sqrt{10-1} = 15$ edges, it has girth at most four. A bipartite graph with five vertices in each part has girth at least four. Five vertices in each part can have up to 25 edges, so we can pick any 20 of these edges to be in the graph and have a graph with 20 edges and girth exactly 4. Therefore, the largest possible girth is 4.

5. Either prove that a graph with an isthmus is not Hamiltonian or else give a counterexample.

Let the isthmus be e and its vertices be x and y. If the graph is Hamiltonian, then it has a Hamiltonian cycle, which includes two paths from x to y with no edges in common (going around the cycle in opposite directions). At most one of these two paths from x to y can use e. The other must be a path from x to y that does not use e. However, if we remove e from the graph, it is disconnected with x and y in opposite parts because e is an isthmus. Therefore there is no path from x to y that doesn't use e, and so the graph is not Hamiltonian.

6. Determine the smallest possible chromatic number of a graph with 9 vertices and 31 edges.

We can compute $\binom{9}{2}_4 = 30$ and $\binom{9}{2}_5 = 32$. By Turan's theorem, a graph with 9 vertices and more than 30 edges must contain a 5-clique. A graph with a 5-clique has chromatic number at least five, as the five vertices of the clique must all be different colors.

The Turan graph $T_{9,5}$ has chromatic number 5, and has 32 edges. If we remove one edge, this does not increase the chromatic number, so it leaves a graph with 9 vertices, 31 edges, and chromatic number at most 5. Since we have seen above that the chromatic number must be at least 5, it is exactly 5, and this is the smallest possible chromatic number for the graph.