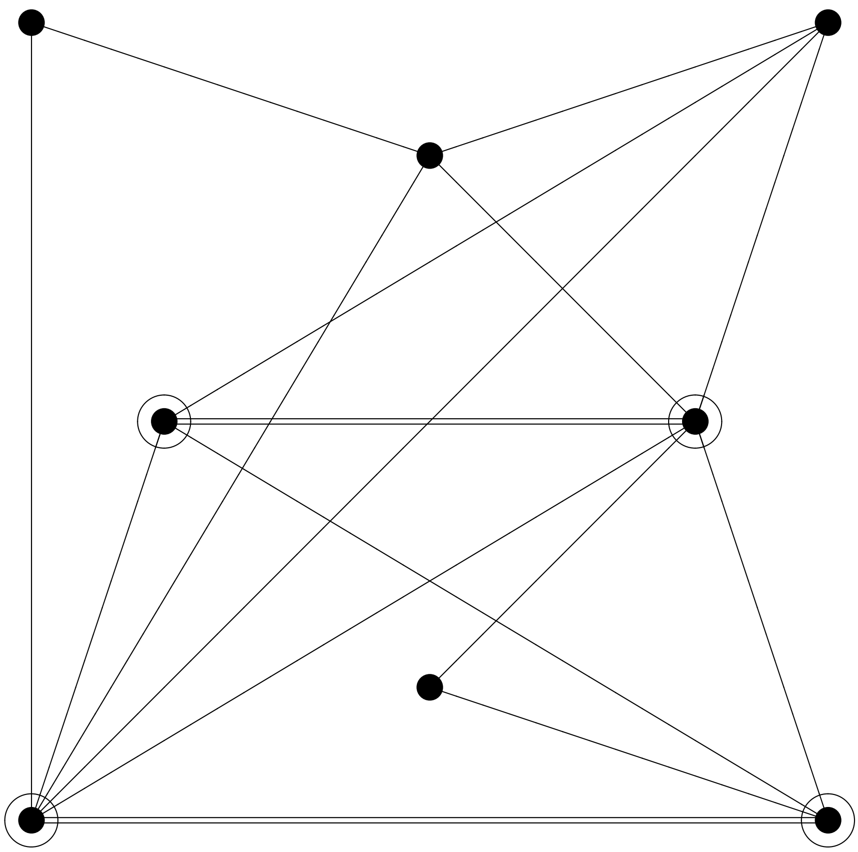
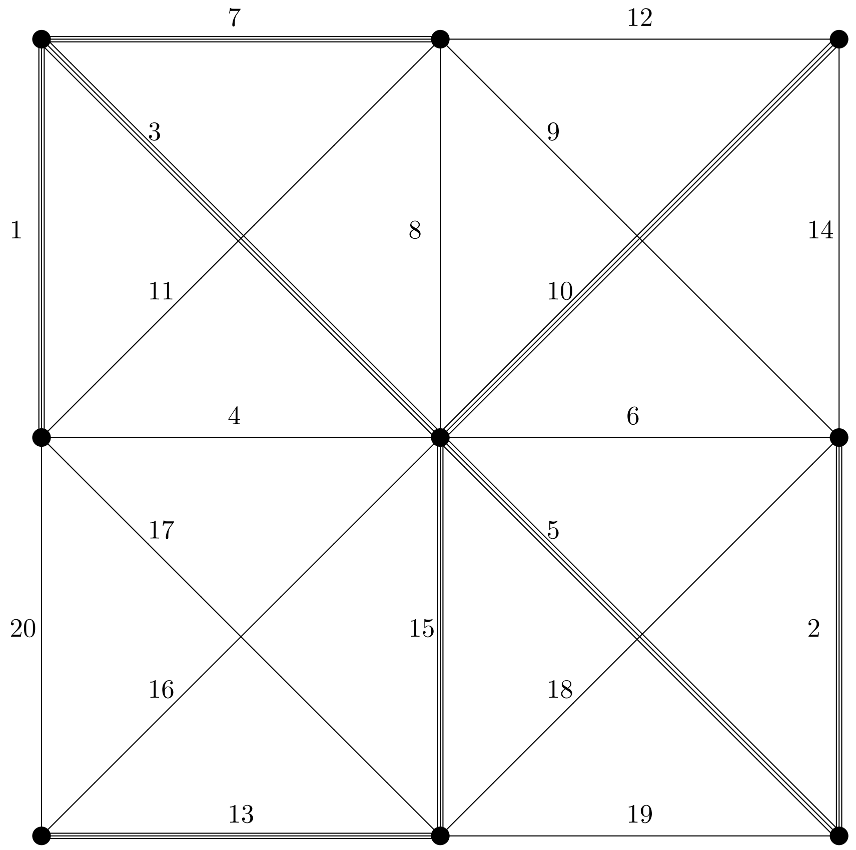
1. The following graph is not Eulerian. Make it into an Eulerian graph by adding as few edges as possible.



A graph is Eulerian if it has an Eulerian circuit, which occurs if the graph is connected and all vertices have even degree. One can check the degrees of the vertices and find that the circled vertices above have odd degree. The way to make them have even degree by adding as few edges as possible is to add an edge to each of the circled vertices and no others. The only way to do this is by adding the two edges with the doubled lines shown above.

2. Find a minimal cost spanning tree for the following weighted graph.



We use the greedy algorithm to add the lowest cost edge at each step that won't create a cycle. The edges added are the tripled ones above.

3. A simple graph G has ten vertices and 39 edges and no clique on r vertices for some natural number r. Determine the smallest possible value of r.

Turán's theorem says that if a graph on 10 vertices has no clique on r vertices, then it has at most $\binom{10}{2}_{r-1}$ edges. It is perhaps easiest to compute the number of edges in $T_{n,r}$ by noting that there are $\binom{10}{2}$ possible edges in total, and counting the number that are missing.

$T_{10,5}$ has five colors, and two vertices of each color. There is one missing edge corresponding to each color, so $\binom{10}{2}_5 = 45 - 5 = 40$. Since $39 < 40$, it is possible for G to have the largest clique of size 5, so $r \leq 6$.

$T_{10,4}$ has four colors, and three vertices each for two of the colors, and two vertices each for the others. The former pair each have 3 missing edges, and the latter each have 1, so $\binom{10}{2}_4 = 45 - 8 = 37$. Since $39 > 37$, G must have a clique of size at least 5, so $r > 5$. Therefore, $r = 6$.

4. Give the smallest upper bound on the Ramsey number $N(4,3,3)$ that you can. You may assume that $N(3,3,3) = 17$ and $N(4,3) = 9$. You are not required to compute $N(4,3,3)$ exactly, but only to give as good of an upper bound as you can.

We are searching for a number of vertices such that, if all edges of a complete graph are colored red, blue, or green, there is no red K_4, blue K_3, or green K_3. Pick a vertex v. If v is incident to at least 17 red edges, then among the 17 vertices at the other end of those edges, there is a monochromatic triangle because $N(3,3,3) = 17$. If it is red, then those three vertices combined with v form a red K_4. Hence, v is incident to at most 16 red edges.

Similarly, if v is incident to at least at least 9 blue edges, then among the nine vertices at the other ends of those edges, there is either a red K_4, a green K_3, or a blue edge, because $N(4,3,2) = N(4,3) = 9$. If it is a blue edge, then those two vertices together with v form a blue triangle. Therefore, v is incident to at most 8 blue edges.

By the same argument, v is incident to at most 8 green edges. As such, v has degree at most 32, so the graph has at most 33 vertices. Therefore, $N(4,3,3) \leq 34$.

Alternatively, you could have used the equation presented in the proof of Ramsey's theorem in class, which is effectively derived by the above equation.

$$\begin{aligned}
N(q_1, q_2, \ldots, q_r) &\leq N(q_1 - 1, q_2, \ldots, q_r) + \ldots + N(q_1, q_2, \ldots, q_r - 1) - r + 2 \\
N(4,3,3) &\leq N(3,3,3) + N(4,2,3) + N(4,3,2) - 3 + 2 \\
&= 17 + 9 + 9 - 3 + 2 \\
&= 34
\end{aligned}$$

While the Ramsey number $N(4,3,3)$ is not known, it is known that $30 \leq N(4,3,3) \leq 32$. Proving an upper bound of 34 was sufficient for full credit on this problem, as tighter bounds are much harder to prove. Proving larger bounds would get you partial credit.

5. How many labeled trees are there on six vertices such that every vertex of the tree has odd degree?

Consider the Prufer codes of such a labeled tree. The degree of each vertex is one greater than the number of the number of times the vertex appears in the first four entries of the second line. As such, these four entries must either all be the same, or have two vertices appear twice each. If all four entries are the same, there are six ways to pick which vertex appears. If two vertices appear twice each, there are $\binom{6}{2} = 15$ ways to pick which two vertices appear, and then $\binom{4}{2} = 6$ ways to pick which vertices appear where, for 90 possibilities. Add these to get 96 total labeled trees.

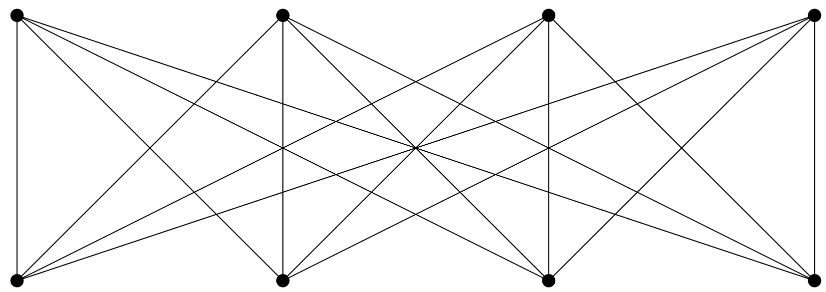
Alternatively, you could note that there are only two trees on six vertices such that every vertex has odd degree up to isomorphism, find the two trees, and then count the number of labeled trees as in the homework problem.

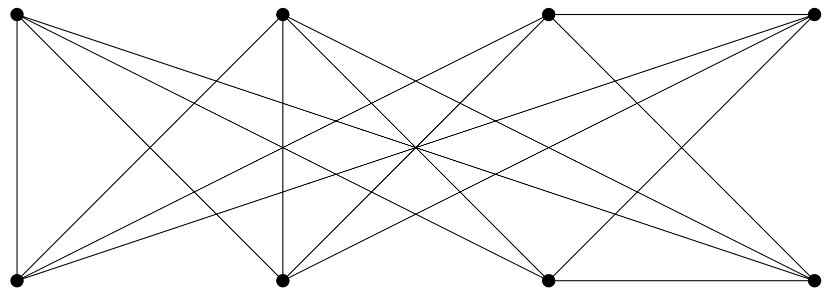
6. Give all non-isomorphic trees on thirteen vertices such that every vertex of the tree has odd degree.

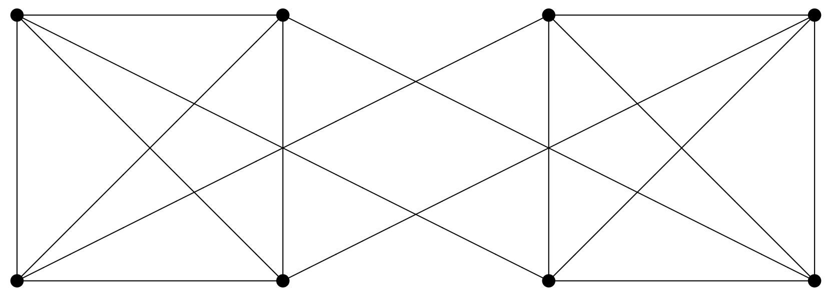
Because the sum of the degrees of the vertices is twice the number of edges, it is even. Thus, the number of vertices of odd degree of a graph is even. In particular, a graph on 13 vertices cannot have all vertices with odd degree, whether it is a tree or otherwise. Therefore, there are no such trees.

7. A graph G is regular of degree four and has eight vertices. Determine all possible values for the chromatic number $\chi(G)$ and give a graph with each possible chromatic number.

In order to have a K_5, the other three vertices would have to each have degree four without being adjacent to any of the first five, which is impossible. Since there is no K_5, by Brooks' theorem, $\chi(G) \leq 4$. Since every vertex has degree four, there is at least one edge, so $\chi(G) \geq 2$. This leaves $\chi(G) = 2, 3, 4$ as the possibilities. The following three graphs have chromatic number 2, 3, and 4, respectively.







The first graph is $K_{4,4}$, and is the unique graph of chromatic number 2 up to isomorphism. The second graph has several triangles, so it has chromatic number at least three. It can be colored with 3 colors by coloring the two far right vertices the same color, and then the rest of the graph is a $K_{3,3}$. The third graph has two K_4s, so it has chromatic number at least 4. There are many possible choices for the graphs of chromatic numbers 3 and 4.