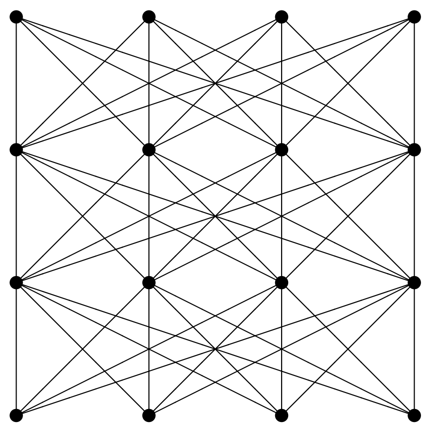
1. A poset P has no chain on five elements and no antichain on five elements. Determine, with proof, the largest possible number of elements in P.

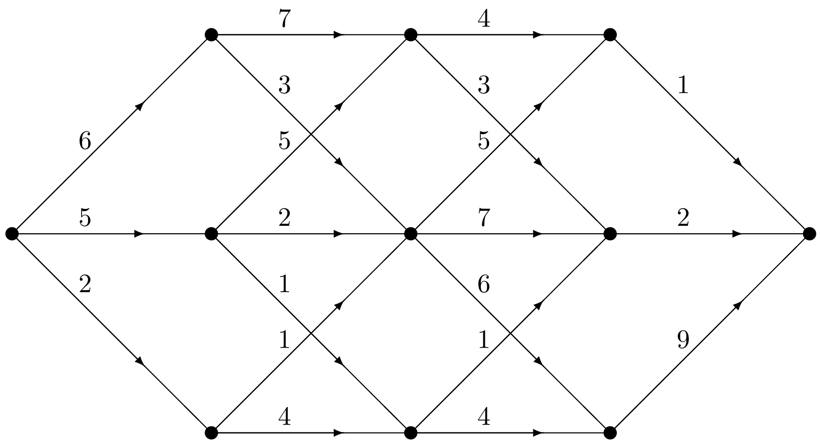
By Mirsky's Theorem, since P has no chain on five elements, P is a union of four antichains. Each of these four antichains has at most four elements, so they have at most 16 elements in total. The following Hasse diagram of a poset shows that it is possible to for P to have exactly 16 elements, so this is the best possible.



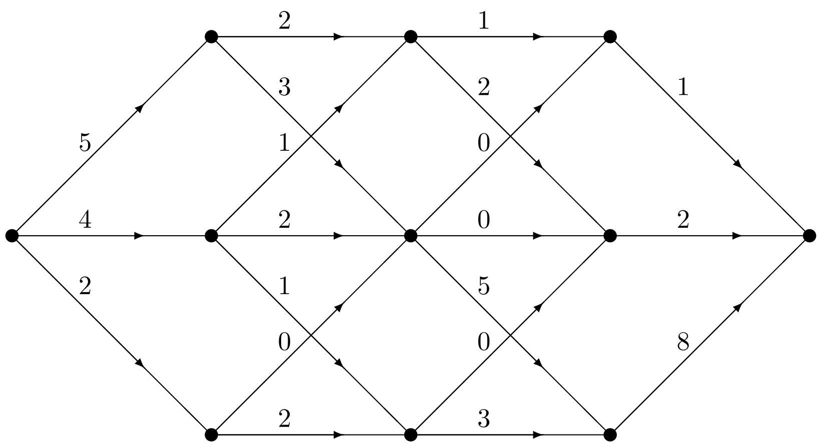
Any two elements of the same column are comparable, so an antichain can have at most one element in each column, and hence at most four elements in total. Any two elements in the same row are not comparable, so a chain can have at most one element in each row, and hence at most four elements in total.

Getting the upper bound of 16 wasn't too hard, but it was essential to show that it was possible to have $|P| = 16$ to get full credit.

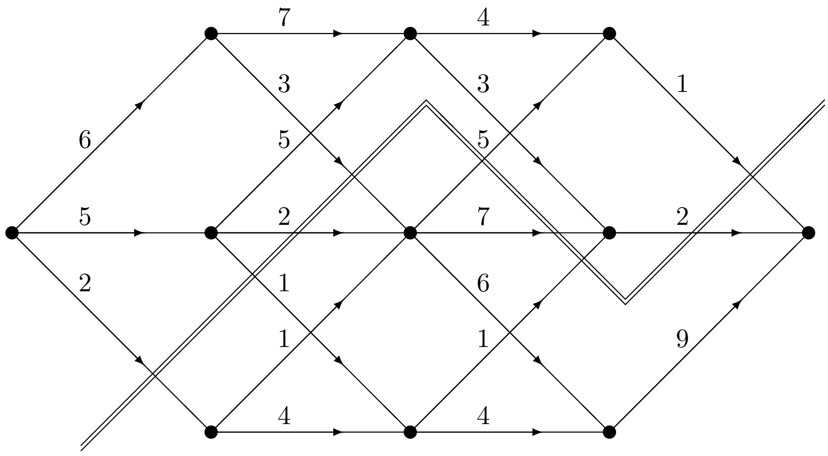
2. Find a minimum cut in the following transportation network. Be sure to prove that it really is a minimum cut. (All numbers give the capacity on the edge just below the number.)



The easiest way to find a minimum cut is to find a maximum flow. By the max flow-min cut theorem, the strength of the maximum flow is the capacity of the minimum cut. We can find a maximum flow by starting with the zero flow and repeatedly finding augmenting paths. One such maximum flow is given below.



This flow has strength 11, so the minimum cut has capacity 11 and is drawn below. We find the minimum cut by saying that all vertices we can reach by an attempt at an augmenting path is on the left side of the cut, and everything else is on the right side of it.



Note that edges of capacities 5, 7, and 1 that cross the cut cross it backwards, so they do not contribute to the capacity of the cut.

3. Bob wants to construct a De Bruijn sequence of order 3 (i.e., all sequences of length 3), but doesn't know how. He knows that it should have four zeroes and four ones, but isn't sure what order to put them in. If he arranges them randomly, what is the probability that he gets a De Bruijn sequence?

By Theorem 8.2, there are $2^{2^{3-1}-3} = 2$ complete cycles on G_3. Each of these has $2^3 = 8$ possible starting points, so there are 16 possible orders of the numbers that give a De Bruijn sequence. There are $\binom{8}{4} = 70$ possible ways to pick which four places out of eight get a zero. Hence, the probability that Bob accidentally comes up with a De Bruijn sequence is $\frac{16}{70} = \frac{8}{35} \approx .2286$.

4. Let A_1, A_2, A_3, and A_4 be subsets of [9] with $|A_1| = 4$, $|A_2| = 5$, $|A_3| = 3$, and $|A_4| = 4$. Determine the smallest number of SDRs that there could be for the sequence of subsets (A_1, A_2, A_3, A_4), and prove that your number is a lower bound for the number of SDRs.

While you are expected to find the largest such lower bound, you do not need to prove that it is the largest lower bound. E.g., if the answer were 42, you would be expected to prove that there must be at least 42 SDRs, but not that it is possible to have exactly 42 SDRs.

First, we note that the sequence of subsets satisfies Hall's condition, as if we only take one, it has size at least $3 > 1$, and if we take two or more, at least one of them has size at least four, so the union has size at least 4, and there are only 4 subsets.

We sort the sets by size to get that we have sets of size 3, 4, 4, and 5. By Theorem 5.3, the number of SDRs is at least $(3-0)_*(4-1)_*(4-2)_*(5-3)_* = (3)(3)(2)(2) = 36$.

Furthermore, it is possible to have exactly 36 SDRs. Let $|A_1| = [4]$, $|A_2| = [5]$, $|A_3| = [3]$, and $|A_4| = [4]$. Pick a representative of A_3 first, then A_4, then A_1, then A_2. There are three ways to pick the representative of A_3. There are four elements of A_4, one of which is already used for A_3, so there are 3 ways to pick the representative of A_4. There are four elements of A_1, two of which are already used, so there are two ways to pick the representative of A_1. There are five elements of A_2, three of which are already in use, so there are two ways to pick the representative of A_2. This gives $3 * 3 * 2 * 2 = 36$ possible SDRs. This paragraph was not necessary for full credit, but I include it to show that 36 is the correct answer.

If you wanted to cite Theorem 5.3, it was necessary to show that the sequence of subsets satisfies Hall's condition. The theorem does not apply if Hall's condition is not satisfied, as it always gives a lower bound of 1 if it applies, and it is possible for a sequence of subsets to have no system of distinct representatives.

5. Let $A_1, A_2, \ldots, A_m$ be subsets of $[n]$ such that $|A_i| \geq k \geq \frac{n}{2}$ for all i and $A_i \cup A_j \neq [n]$ for all $i \neq j$. Show that $m \leq \binom{n-1}{k}$.

Let $B_i = [n] - A_i$ for each i. Then B_i are subsets of $[n]$. We have $|B_i| = n - |A_i| \leq n - k \leq n - \frac{n}{2} = \frac{n}{2}$. That $A_i \cup A_j \neq [n]$ means that there is some $w \in [n]$ in neither A_i nor A_j, so it is in both B_i and B_j. Hence, $w \in B_i \cap B_j$, and so $B_i \cap B_j \neq \emptyset$. By Theorem 6.5 (a generalization of Erdös-Ko-Rado), we have $m \leq \binom{n-1}{n-k-1} = \binom{n-1}{k}$.

6. Prove Hall's Theorem, that is, Theorem 5.1: given a bipartite graph G with parts X and Y, there is a complete matching from X to Y if and only if $|\Gamma(A)| \geq |A|$ for every $A \subseteq X$.

You may cite any theorem from the book except for Theorem 5.1 itself. You may not cite homework problem 7D without proving it.

You had at least four choices here. One is to reproduce the original proof of Theorem 5.1, as given both in the book and in class. Another, and perhaps the simplest, was to cite Theorem 5.3 and explain why the formula always evaluates to at least one, and how SDRs of sequences of subsets correspond to complete matchings. Third, you could have cited Dilworth's Theorem (Theorem 6.1) and derived Hall's Theorem from it, as given both in the book and in class. Fourth, you could have reproduced your answer to homework problem 7D, to derive Hall's Theorem from the Max Flow-Min Cut Theorem and its integer flow version (Theorems 7.1 and 7.2). Any of these would earn full credit. All are available in the book and/or on the course web page, so I won't reproduce them here.