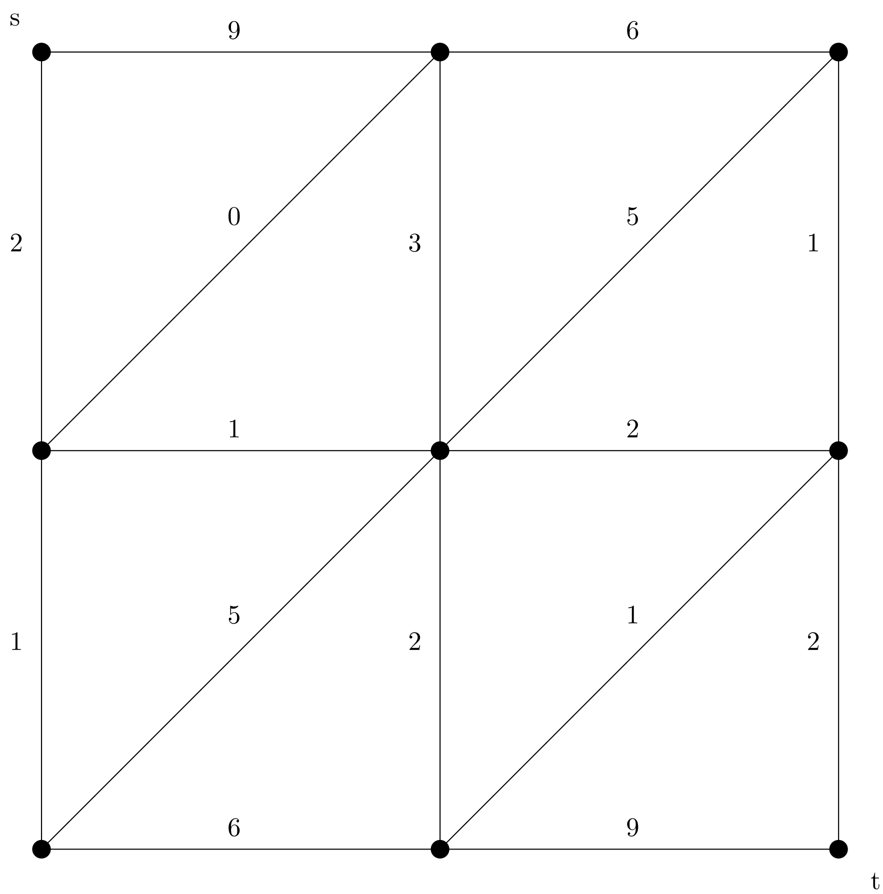
1. Find a maximum flow and a minimum cut for the following transportation network. Interpret each edge shown as being an edge in each direction with the listed capacity.



We start with the zero flow, find a special path, and add as much capacity as possible along that path. We repeat until there is no special path anymore. If we label the vertices as $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then one possible flow comes from the path a-b-c-e-g-h-i with flow 5, a-b-e-h-i with flow 2, a-b-d-e-f-h-i with flow 2, a-b-c-f-i with flow 1, and a-d-g-h-i with flow 1. This gives the above flow, with all flows going downward, except horizontal ones, which flow to the right.

The only minimum cut consists of a, b, c, d, and e on one side and f, g, h, and i on the other side.

This problem ended up being harder than I expected. Several students didn't seem to know what the capacity of a cut is. The capacity of a cut (X, Y) is the sum of the capacities of all edges from X to Y. It is not merely the sum of the flows on the edges, as the capacity of a cut is a property of the transportation network, not of any particular flow on it. Edges from Y to X are irrelevant to the capacity of the cut.

I expected to see a final answer (i.e., here's a particular maximum flow, not just the strength of the flow), as well as some work leading to it (e.g., what the network looks like after each path, or at least a list of the paths used to generate the final flow). Some students had the former without the latter, and some, oddly, had the latter without the former.

2. Give a symmetric chain decomposition of B_4.

We use the inductive procedure given in the book.

$B_1 : \{\emptyset, \{1\}\}$

$B_2 : \{\{1\}\}$
$\{\emptyset; \{2\}; \{1,2\}\}$

$B_3 : \{\{1\}; \{1,3\}\}$
$\{\{2\}; \{1,2\}\}$
$\{\emptyset; \{3\}; \{2,3\}; \{1,2,3\}\}$

$B_4 : \{\{1,3\}\}$
$\{\{1\}; \{1,4\}; \{1,3,4\}\}$
$\{\{1,2\}\}$
$\{\{2\}; \{2,4\}; \{1,2,4\}\}$
$\{\{3\}; \{2,3\}; \{1,2,3\}\}$
$\{\emptyset; \{4\}; \{3,4\}; \{2,3,4\}; \{1,2,3,4\}\}$

This problem was pretty easy. The intended solution to this problem was to use the algorithm in the book, though the problem didn't explicitly ask for that. Some students drew the poset and picked out symmetric chains that way, starting with the longest ones, and working down to smaller ones that are easier to build from leftover sets. This was also acceptable.

3. Give a De Bruijn sequence that gives all possible four digit sequences of 0's and 1's.

We make a directed graph with eight vertices, one for each three digit sequence. We have an edge go from x to y if and only if the last two digits of x are the first two digits of y in the same order. A De Bruijn sequence corresponds to an Eulerian cycle. There are 16 possible sequences. One of them is 0000111101100101. This particular sequence is the Ford sequence as described in problem 8D.

This problem was also easy. The intended solution was to draw a graph as discussed both in class and in the book and find an Eulerian cycle. Several students used the approach of problem 8D to find the Ford sequence, which also worked.

4. Give a matrix such that all entries are integers, each entry differs from the following matrix by less than 1, the sum of the entries in each column differs from the following matrix by less than 1, and the sum of the entries in each row differs from the following matrix by less than 1.

$$\begin{bmatrix} 4.2 & 3.6 & -1.8 \\ 5.3 & 0.3 & 2.4 \\ -1.5 & -3.9 & 6.4 \end{bmatrix}$$

We can create a circulation as in the proof of Theorem 7.5 and increase flow in a particular direction on various cycles as in the proof of Theorem 7.4 to give more edges an integer values for their flows. Once all edges have integer flows, the flows on the relevant edges give us the entries of the matrix. The particular matrix will depend on the cycles chosen, so there are a number of possible answers. It's hard to typeset the graph, but one possibility is that the values for the matrix after each step is this.

$$\begin{bmatrix} 4 & 3.8 & -1.8 \\ 5.5 & 0.1 & 2.4 \\ -1.5 & -3.9 & 6.4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3.9 & -1.9 \\ 5.5 & 0 & 2.5 \\ -1.5 & -3.9 & 6.4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -2 \\ 5.5 & 0 & 2.5 \\ -1.5 & -4 & 6.5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -2 \\ 6 & 0 & 2 \\ -2 & -4 & 7 \end{bmatrix}$$

In writing this problem, I worried that students may be able to solve the problem without knowing the intended method. That's about how it happened, and the problem ended up being pretty easy for that reason. For people who had a correct answer, I didn't dock any points for not showing work, since this is the sort of problem that one realistically could do by trial and error.

5. Say that a bipartite graph with vertices $X = \{x_1, x_2, \dots, x_m\}$ in one part and $Y = \{y_1, y_2, \dots, y_n\}$ in the other part is *shifted* if, for every $a \leq b$ and $c \leq d$, if $\{x_b, y_d\}$ is an edge, then $\{x_a, y_c\}$ is also an edge. Show that if a shifted bipartite graph has a perfect matching of X into Y, then $\{x_m, y_1\}, \{x_{m-1}, y_2\}, \dots, \{x_1, y_m\}$ is such a perfect matching.

Solution #1: Suppose that the given proposed perfect matching does not work. This means that for some i, the edge $\{x_i, y_{m+1-i}\}$ must not be in the graph. Because the graph is shifted, for any $j \geq i$ and $k \geq m+1-i$, $\{x_j, y_k\}$ is not an edge of the graph. If we let $A = \{x_i, x_{i+1}, \dots, x_m\}$, then $\Gamma(A) \subset \{y_1, y_2, \dots, y_{m-i}\}$. This gives us $|A| = m - i + 1 > m - i \geq |\Gamma(A)|$, so by Theorem 5.1 (Philip Hall's Theorem), there is no perfect matching of X into Y.

Solution #2: Suppose that there is a perfect matching. We wish to show that, for each i, the edge $\{x_i, y_{m+1-i}\}$ is in the graph. By Theorem 5.1, for any $A \subset X$, $|\Gamma(A)| \geq |A|$. Let $A = \{x_i, x_{i+1}, \dots, x_m\}$. Then $|A| = m + 1 - i$, so $|\Gamma(A)| \geq m + 1 - i$. Since there are at least $m + 1 - i$ vertices in $\Gamma(A)$, all of which are in Y, $y_j \in \Gamma(A)$ for some $j \geq m + 1 - i$. Let x_k be a vertex of A to which y_j is adjacent. Since $x_k \in A$, $k \geq i$. Thus, we have that $\{x_k, y_j\}$ is an edge with $k \geq i$ and $j \geq m + 1 - i$. Since the graph is shifted, this means that $\{x_i, y_{m+1-i}\}$ is an edge of the graph.

The two solutions above are pretty similar. The first is what I typed before the exam, and the second is closer to what a lot of students tried, and essentially the way to fix the common incorrect argument. Any attempt at an induction argument was pretty hopeless. A number of students tried to start by saying that x_m is adjacent to y_1, and then assuming that if there is a perfect matching, then there must be a perfect matching that uses this particular edge. For non-shifted graphs, this would be false. As expected, this problem was fairly difficult.

I intended to say a complete matching, not a perfect one, which is why I didn't specify that $m = n$. Thankfully, that didn't affect the sense of the problem.

Some mathematicians might object to me using the definition given in the problem as the definition of "shifted", and what I stated in this problem isn't quite what the word usually means, though it is pretty close.

6. Let P be a poset whose elements are subsets of [9] of size three, with order defined such that $a \leq b$ if the smallest element of a is less than or equal to the smallest element of b, the second smallest element of a is less than or equal to the second smallest of b, and the largest element of a is less than or equal to the largest element of b. For example, $\{2, 6, 3\} \leq \{9, 5, 3\}$ because $2 \leq 3$, $3 \leq 5$, and $6 \leq 9$. As another example, $\{1, 3, 7\}$ is not comparable to $\{2, 4, 6\}$ because $3 < 4$ but $7 > 6$. If we wish to partition P into as few antichains as possible, determine the minimum number of antichains needed.

By Theorem 6.2 (Mirsky's Theorem), the minimum number of antichains needed is the maximum length of a chain. If $a < b$ then the sum of the numbers in a must be less than the sum in b. The sum of the numbers in any element of P is an integer, so the sum for b must exceed that of a by at least 1. The minimal element of the poset is $\{1, 2, 3\}$ with sum 6, while the maximal element is $\{7, 8, 9\}$ with sum 24, so a chain can have length at most 19. We can get a chain of length 19 by starting with $\{1, 2, 3\}$ and adding 1 to the last element repeatedly until it is a 9, then 1 to the middle element repeatedly until it reaches 8, and then 1 to the first element repeatedly. We can partition P into 19 antichains, with an antichain for elements of sum i for each of $6 \leq i \leq 24$.

This problem was one of the hardest ones on the exam, and I didn't give anyone full credit for the problem. A complete solution required finding a chain of length 19 (proving that the answer is at least 19), and then either partitioning the poset into 19 antichains or proving that there is no longer chain and citing Mirsky's theorem (proving that the answer is at most 19).

Finding a chain of length 19 and then asserting that there is no longer chain does not prove that there isn't a longer one. Showing that it isn't possible to add more elements to a particular chain does not prove that there can't be some other chain that is longer. For example, if we were to add a new element x to the poset with $\{1, 2, 3\} < x < \{7, 8, 9\}$ and x not comparable to any other elements, then this chain would have length 3, and no other elements could be added to it, but the poset would still have other, longer chains. The proper method is to refer to the sum of the numbers in each element of the poset.

Some people tried to make multisets with repeated elements, such as $\{1, 1, 1\}$ as an element of the poset. Some also considered $\{1, 2, 3\}$ and $\{2, 1, 3\}$ as distinct elements, even though they gave the same set. Both of these were rather bad, though analogous problems for multisets or ordered triples of numbers had a very similar solution, so it was still possible to get a lot of partial credit.

Also, since $|P| = \binom{9}{3} = 84$, one could trivially partition P into 84 antichains by making each element its own antichain. Without doing much work, it should be clear that the final answer is not larger than this.