

Math 4410
Fall 2010
Exam 3

Name:

Directions:

Complete all six questions.

Show your work. A correct answer without any scratch work or justification may not receive much credit.

You may not use any notes, calculators, or other electronic devices.

You have 90 minutes.

Problem 1: _____ / 10

Problem 2: _____ / 10

Problem 3: _____ / 10

Problem 4: _____ / 10

Problem 5: _____ / 10

Problem 6: _____ / 10

Total: _____ / 60

1. How many ways are there to put n indistinguishable balls into k distinguishable boxes such that there are at least three balls in each box?

We start by putting 3 balls in each box. Then the problem is one of putting $n - 3k$ balls into k boxes. By Theorem 13.1, the answer to this is

$$\binom{n - 3k + k - 1}{k - 1} = \binom{n - 2k - 1}{k - 1}.$$

2. Find an exponential generating function for the number of regular graphs of degree 1 on n vertices.

In order for every vertex to have degree 1, the graph must consist of $\frac{n}{2}$ edges, no two of which have a common endpoint. This is equivalent to breaking the set of vertices into parts, and then requiring each part to have two vertices so that we can apply a vertex to it. This is only possible if each part has exactly two vertices. The exponential generating function for each part for the requirement that it has two vertices is $\frac{x^2}{2}$. Therefore, by Theorem 14.2, the answer is

$$e^{\frac{x^2}{2}}.$$

Another approach is to let a_n be the number of such graphs on n vertices. If we have such a graph, then vertex n must be adjacent to exactly one other. There are $n - 1$ ways to pick the other vertex. If we remove vertex n and its neighbor, then we must have a regular graph of degree 1 on $n - 2$ vertices. There are a_{n-2} ways to do this. Hence, $a_n = (n - 1)a_{n-2}$.

The exponential generating function is $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. We can compute $f'(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$. Since $a_1 = 0$, we can drop the $n = 0$ term and get $f'(x) = \sum_{n=1}^{\infty} a_{n+1} \frac{x^n}{n!}$. We apply the recursion to get

$$f'(x) = \sum_{n=1}^{\infty} n a_{n-1} \frac{x^n}{n!} = \sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{(n-1)!} = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} = x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = x f(x).$$

If $f(x) = e^{g(x)}$, then $f'(x) = g'(x)e^{g(x)} = g'(x)f(x)$. We have $g'(x) = x$, from which $g(x) = \frac{x^2}{2} + C$. Therefore,

$$f(x) = e^{\frac{x^2}{2} + C} = e^C e^{\frac{x^2}{2}}.$$

The constant term here is $1 = a_0 = e^C e^{\frac{0^2}{2}} = e^C$, so we get an answer of

$$f(x) = e^{\frac{x^2}{2}}.$$

A third approach is to start by computing a_n directly. There must be $\frac{n}{2}$ edges in order for the degree sum to be n , which is impossible if n is odd. If n is even, then pick a vertex, and there are $n - 1$ possible choices for its neighbor. Pick another vertex that hasn't already been used, and there are $n - 3$ possible choices for its neighbor. Continue this until we have a perfect matching. There are $(n - 1)(n - 3)(n - 5) \dots 1$ possible choices at the various steps, so

$$a_n = (n - 1)(n - 3)(n - 5) \dots 1 = \frac{n(n - 1)(n - 2) \dots (1)}{(2 \frac{n}{2})(2(\frac{n}{2} - 1)) \dots (1)} = \frac{n!}{2^{\frac{n}{2}} (\frac{n}{2})!}.$$

Equivalently,

$$a_{2n} = \frac{(2n)!}{2^n (n!)}.$$

The odd terms are all zero, so we can skip them and get an exponential generating function of

$$f(x) = \sum_{n=0}^{\infty} a_{2n} \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!) (2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n (n!)}.$$

Let $y = x^2$. This gives

$$f(x) = \sum_{n=0}^{\infty} \frac{y^n}{2^n (n!)} = \sum_{n=0}^{\infty} \frac{\left(\frac{y}{2}\right)^n}{(n!)} = e^{\frac{y}{2}} = e^{\frac{x^2}{2}}.$$

3. How many ways are there to place numbers from $[n]$ in four positions on a circle such that no two consecutive places get the same number? Non-consecutive positions on the circle can have the same number. We do not regard two arrangements as the same if one can be obtained from the other by rotation.

We use inclusion-exclusion. There are n ways to pick the number for each position, so there are n^4 ways to place the numbers. There are 4 ways to pick a pair of adjacent positions to be the same, n ways to pick the number for both of those positions, and n ways to pick the numbers for each of the other two positions, so for each way to pick a pair of adjacent positions, there are n^3 arrangements that give the same number in the two chosen positions.

There are 6 ways to pick two pairs of adjacent positions on the circle. If we want the numbers for the chosen pairs to be the same, then there are two ways to pick the pairs opposite each other so that there are two of one number and two of another number. There are n ways to pick each number, and so n^2 ways to pick the numbers. There are 4 ways to pick the pairs such that they require three numbers to be the same. In this case, there are n ways to pick the common number for three positions and n ways to pick the number for the fourth position, so there are again n^2 ways to place the numbers.

There are 4 ways to pick three pairs of adjacent positions on the circle. In all of them, if we require both numbers in a pair to be the same, it forces all four numbers on the circle to be the same. There are n ways to pick the common number. There is 1 way to pick all four pairs of adjacent positions, and this again forces all four numbers on the circle to be the same, so we again have n choices.

Therefore, by the principle of inclusion-exclusion, the number of ways for no pair of adjacent positions to have the same number is

$$n^4 - 4n^3 + 6n^2 - 4n + n = n^4 - 4n^3 + 6n^2 - 3n = (n-1)^4 + (n-1) = n(n-1)(n^2 - 3n + 3).$$

This last factorization implies another approach. We can start by picking a position on the circle and putting a number in it. We have n choices. We can pick a position adjacent to it and put a number in this one. Here, we have $n-1$ choices, as it cannot be the same as the first number.

Now the question is how many ways we can place the remaining two numbers. One way to be safe is to pick two numbers not previously used. This leaves $n-2$ ways to pick the number for the first remaining position and $n-3$ for the second, for $(n-2)(n-3) = n^2 - 5n + 6$ choices.

We still need to count the ways to pick one of the last two positions having the same number as the position opposite to it. If the number opposite the first position is the same as in the first position, then there is 1 way to pick this number, and $n-1$ ways to pick the remaining position. Likewise, if the number opposite the second position is the same as the second position, there is 1 way to pick this number, and $n-1$ ways to pick the remaining position. This double counts the possibility of both of the last two positions having the same number

as the position opposite to them, and there is 1 way for this to happen, so it must be subtracted off. Thus, we have an additional $(n-1)+(n-1)-1 = 2n-3$ arrangements.

Given the first two numbers, there are $(n^2 - 5n + 6) + (2n - 3) = n^2 - 3n + 3$ ways to pick the remaining numbers. There are $n(n-1)$ ways to pick the first two, so the total number of possible arrangements is

$$n(n-1)(n^2 - 3n + 3) = n^4 - 4n^3 + 6n^2 - 3n.$$

A third approach is a direct count. Let the four positions on the circle be $a, b, c,$ and $d,$ in that order, with d adjacent to a at the end of the circle. We could have a and c be the same, or b and d be the same, but all other possible pairs are distinct.

If all four numbers are distinct, then there are n choices for $a,$ $n-1$ choices for $b,$ $n-2$ choices for $c,$ and $n-3$ choices for $d,$ for $n(n-1)(n-2)(n-3)$ choices.

If $a = c$ are the same and $b \neq d,$ then there are n choices for $a = c,$ $n-1$ choices for $b,$ and then $n-2$ choices for $d,$ for $n(n-1)(n-2)$ arrangements. Similarly, there are $n(n-1)(n-2)$ arrangements for if $b = d$ and $a \neq c.$

If $a = c$ and $b = d,$ then there are n choices for $a = c$ and then $n-1$ choices for $b = d,$ for $n(n-1)$ choices.

We add these to get

$$n(n-1)(n-2)(n-3) + 2n(n-1)(n-2) + n(n-1) = n(n-1)((n-2)(n-3) + 2(n-2) + 1) = n(n-1)(n^2 - 3n + 3).$$

Having any of $n^4 - 4n^3 + 6n^2 - 3n,$ $(n-1)^4 + (n-1),$ or $n(n-1)(n^2 - 3n + 3)$ was an acceptable final answer here, as I wouldn't regard any of them as obviously simpler than the others.

This was originally intended as an inclusion-exclusion problem, and most students tried some sort of inclusion-exclusion approach.

4. Show that the number of partitions of n with all parts of size at least k is equal to the number of partitions of n for which the k largest parts are all the same size.

To say that all parts are of size at least k means that every row of the corresponding Ferrers diagram has at least k blocks. This means that the first k columns go all the way down to the bottom row, and hence all have just as many blocks. Therefore, the first k rows in the conjugate partition all have the same length. This forces the k largest parts in the conjugate partition to all be the same size. As all of these steps work in reverse, taking the conjugate Ferrers diagram gives a bijection between the two types of partitions of n specified in the statement of the problem.

5. How many ways are there to put 7 distinguishable balls into 4 indistinguishable boxes?

This is closely related to the Stirling numbers of the second kind. $S(n, k)$ is the number of ways to put n distinguishable balls into k indistinguishable boxes with no box empty. If we allow empty boxes, we sum over how many possible boxes could be used to get an answer of

$$\sum_{i=1}^k S(n, i).$$

If we plug in $n = 7$ and $k = 4$, we get

$$\sum_{i=1}^4 S(7, i).$$

Thus, we merely need to compute some Stirling numbers. We have a recursion $S(n, k) = kS(n-1, k) + S(n-1, k-1)$. We know initial conditions of $S(n, 1) = 1$ because there is only one way to put all of the balls in the same box, and $S(n, n) = 1$, as if each ball has its own box, all arrangements look the same, because we can't tell the difference between the boxes. Thus, we use the recursion to make a table.

n	k	1	2	3	4
1		1			
2		1	1		
3		1	3	1	
4		1	7	6	1
5		1	15	25	10
6		1	31	90	65
7		1	63	301	350

Therefore, the answer is

$$S(7, 1) + S(7, 2) + S(7, 3) + S(7, 4) = 1 + 63 + 301 + 350 = 715.$$

6. Find an ordinary generating function for the sequence given by $a_0 = a_1 = 0$, $a_2 = 1$, and $a_{n+1} = a_{n-1} + a_{n-2}$ for $n \geq 2$.

We can compute

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} a_n x^n \\
 &= x^2 + \sum_{n=3}^{\infty} a_n x^n \\
 &= x^2 + \sum_{n=0}^{\infty} a_{n+3} x^{n+3} \\
 &= x^2 + \sum_{n=0}^{\infty} (a_{n+1} + a_n) x^{n+3} \\
 &= x^2 + \sum_{n=0}^{\infty} a_{n+1} x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+3} \\
 &= x^2 + x^2 \sum_{n=1}^{\infty} a_n x^n + x^3 \sum_{n=0}^{\infty} a_n x^n \\
 &= x^2 + x^2 \sum_{n=0}^{\infty} a_n x^n + x^3 \sum_{n=0}^{\infty} a_n x^n \\
 &= x^2 + x^2 f(x) + x^3 f(x).
 \end{aligned}$$

We rearrange this to get $f(x)(1 - x^2 - x^3) = x^2$, from which

$$f(x) = \frac{x^2}{1 - x^2 - x^3}.$$