1. Give an ordinary generating function for the sequence a_n defined by $a_0 = 1$, $a_1 = 3$, and $a_n = a_{n-1} + 2a_{n-2}$ for all $n \geq 2$.

We compute

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + 3x + \sum_{n=2}^{\infty} a_n x^n \\
&= 1 + 3x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2}) x^n \\
&= 1 + 3x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} 2a_{n-2} x^n \\
&= 1 + 3x + x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + 2x + x \sum_{n=0}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + 2x + x f(x) + 2x^2 f(x) \\
(1 - x - 2x^2) f(x) &= 1 + 2x \\
f(x) &= \frac{1 + 2x}{1 - x - 2x^2}
\end{aligned}$$

2. For any positive integer n, show that there is some value of c (which can depend on n) such that for all $k \geq c$, $p_k(n+k) = p_c(n+c)$. Also find the minimum such value of c.

$p_k(n+k)$ is the number of ways to put $n+k$ indistinguishable balls into k indistinguishable boxes such that no box is empty. If we remove one ball from each box, it is the number of ways to put n balls into k boxes, with some boxes allowed to be empty. If $k > n$, we must leave at least $k-n$ boxes empty, and because the boxes are indistinguishable, removing these boxes that must be empty does not change the number of arrangements. Therefore, for all $k \geq n$, $p_k(n+k) = p_n(2n)$. Furthermore, n is the minimum possible such value of c, as if if $c < n$, then $p_c(n+c)$ does not count the possibility of each extra ball having its own box, so we lose at least one of the arrangements that would be counted in $p_n(2n)$.

3. How many positive integers are there that are factors of at least one of $2^4 3^7$, $3^5 5^9$, and $2^8 5^5$?

We use inclusion-exclusion. Any factor of $2^4 3^7$ must be of the form $2^m 3^n$ for $m \leq 4$ and $n \leq 7$. There are 5 possible choices for m and 8 possible choices for n, so there are 40 factors of $2^4 3^7$. Similarly, there are 60 factors of $3^5 5^9$ and 54 factors of $2^8 5^5$.

Any number that is a factor of both $2^4 3^7$ $3^5 5^9$ must be a factor of their greatest common divisor, which is 3^5. There are 6 such numbers. Similarly, there are 6 numbers that divide both $3^5 5^9$ and $2^8 5^5$, and five that divide both $2^4 3^7$ and $2^8 5^5$. Finally, 1 is the unique common divisor of all three numbers. Therefore, by inclusion-exclusion, the number of factors of at least one such number is

$$40 + 60 + 54 - 6 - 6 - 5 + 1 = 138.$$

4. Give a formula for the Stirling number of the second kind $S(n, 2)$.

The Stirling numbers of the second kind are the ways to put distinguishable balls into indistinguishable boxes with no box empty. There are 2^n possible ways to place the n balls into two distinguishable boxes. Two of these put all of the balls into the same box, so there are $2^n - 2$ arrangements for distinguishable boxes. Divide by 2! ways to rearrange the boxes to get $2^{n-1} - 1$ ways to place the balls into indistinguishable boxes.

5. Give an exponential generating function for the sequence a_n defined by $a_0 = a_1 = 1$ and $a_{n+1} = a_n + n(n-1)a_{n-1}$ for all $n \geq 1$.

We compute

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \\
f'(x) &= \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} (a_n + n(n-1)a_{n-1}) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} n(n-1)a_{n-1} \frac{x^n}{n!} \\
&= f(x) + \sum_{n=2}^{\infty} a_{n-1} \frac{x^n}{(n-2)!} \\
&= f(x) + x^2 \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} \\
&= f(x) + x^2 f'(x).
\end{aligned}$$

This gives us $(1-x^2)f'(x) = f(x)$, or equivalently, $f'(x) = \frac{1}{1-x^2} f(x)$. If $f(x) = e^{g(x)}$, then $f'(x) = g'(x)e^{g(x)} = g'(x)f(x)$, so we have $g'(x) = \frac{1}{1-x^2}$. Then

$$\begin{aligned}
g(x) &= \int g'(x)\ dx \\
&= \int \frac{1}{1-x^2}\ dx \\
&= \int \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)\ dx \\
&= \frac{1}{2}\Big(\ln(1+x) - \ln(1-x)\Big) + C \\
&= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) + C \\
&= \ln\left(\sqrt{\frac{1+x}{1-x}}\right) + C.
\end{aligned}$$

From this, $f(x) = e^{g(x)} = \sqrt{\frac{1+x}{1-x}} e^C$. If $x = 0$, then $f(0) = a_0 = 1$. The function gives $f(0) = \sqrt{\frac{1+0}{1-0}} e^C = e^C$, so $e^C = 1$. Therefore, $f(x) = \sqrt{\frac{1+x}{1-x}}$.

6. Let P be the set of partitions of 150 such that for all $k \geq 1$, if there is a part of size $k+1$, then there is at least one part of size k. Show that the number of partitions in P for which the largest part is even is equal to the number of partitions in P for which the largest part is odd.

Let P^* be the set of conjugates of partitions in P. If a partition is in P, then if there is a $(k+1)$-th column of the Ferrers diagram, then the k-th column is longer, because there is a part of size k. Therefore, no two columns have the same length. Columns of a partition are rows of the Ferrers diagram in its conjugate partition, so for every partition in P^*, no two rows are the same length. This means that all partitions in P^* have no two parts of the same size. These steps work in reverse also, so P^* is the set of partitions of 150 with no two parts equal.

The size of the largest part of a partition is the number of parts of its conjugate partition. Therefore, the problem asks us to show that the number of partitions in P^* with an odd number of parts is equal to that for an even number of parts. If 150 is not a pentagonal number, then this was a theorem in class, and is essentially Theorem 15.5 in the book. We can check $\omega(10) = 145$, $\omega(11) = 176$, $\omega(-10) = 155$, and $\omega(-9) = 126$, so if $\omega(m) = 150$, then $10 < m < 11$ if $m > 0$ and $-10 < m < -9$ if $m < 0$, both of which are obviously impossible.