Turán's theorem, Zykov's theorem, and the $\binom{n}{k}_r$ notation

A *Turán graph* $T_{n,r}$ has n vertices divided into r parts as evenly as possible. That is, all parts have either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$ vertices. Two vertices in the graph are adjacent exactly if they are not in the same part.

You can think of the r parts as being r colors, and then two vertices are adjacent if and only if they are different colors. The graph $T_{n,r}$ is r-colorable, as we can give each part its own color.

A *clique* is a complete subgraph. A k-clique is a clique with k vertices. That is, it is a set of k vertices such that every pair of vertices is connected by an edge. A 1-clique is a vertex and a 2-clique is an edge. 3-cliques are sometimes called triangles.

We define $\binom{n}{k}_r$ to be the number of k-cliques in the graph $T_{n,r}$. Thus, $\binom{n}{1}_r$ is the number of vertices, which is n. $\binom{n}{2}_r$ is the number of edges. If $r \geq n$, then every vertex has its own color, so the graph $T_{n,r}$ is the complete graph K_n, and $\binom{n}{k}_r = \binom{n}{k}$. In general, $\binom{n}{k}_r \leq \binom{n}{k}$.

$\binom{n}{k}_r$ can be somewhat awkward to compute numerically. If the graph is small enough, one can count cliques pretty directly. One quick approximation that is pretty good for $n \gg r$ is

$$\binom{n}{k}_r \approx \binom{r}{k}\left(\frac{n}{k}\right)^k.$$

In general,

$$\binom{n}{k}_r \leq \binom{r}{k}\left(\frac{n}{k}\right)^k,$$

and equality only holds if n is an integer multiple of k.

If it is larger, then we can set $p = \lceil \frac{n}{r} \rceil - 1$, and $q = n - pr$. This makes it so that p and q are the unique integers such that $n = pr + q$ and $1 \leq q \leq r$. With this definition, the known formulas and recursions include these.

$$\begin{aligned}
\binom{n}{k}_r &= \sum_{i=0}^{q} \binom{q}{i}\binom{r-i}{k-i} p^{k-i} \\
\binom{n}{k}_r &= \sum_{i=0}^{q} \binom{q}{i}\binom{r-q}{k-i} (p+1)^i p^{k-i} \\
\binom{n}{k}_r &= \binom{n}{k} + \sum_{i=1}^{k} (-1)^i \left(q\binom{p+1}{i}\binom{n-p-1}{k-i} + (r-q)\binom{p}{i}\binom{n-p}{k-i} \right) \\
\binom{n}{k}_r &= \binom{n-1}{k}_r + \binom{n-p-1}{k-1}_{r-1} \\
k\binom{n}{k}_r &= (p+1)q\binom{n-p-1}{k-1}_{r-1} + p(r-q)\binom{n-p}{k-1}_{r-1}
\end{aligned}$$

With this notation, Turán's theorem [1] is easy to state.

Theorem 1 (Turán) *A graph with n vertices and no clique on $r+1$ vertices has at most $\binom{n}{2}_r$ edges.*

I find this much easier to remember than the messy formula in the book. Mantel's theorem is the case where $r = 2$. Zykov's theorem [2] generalizes Turán's theorem.

Theorem 2 (Zykov) *A graph with n vertices and no clique on $r+1$ vertices has at most $\binom{n}{k}_r$ k-cliques.*

Turán's theorem is the case where $k = 2$.

If the book tried to give an explicit formula without discussing Turán graphs, Zykov's theorem would be a horrendous mess. Hence, the book doesn't include it. But it's actually a nice theorem, and easy to state and remember if you have the right definitions in place.

References

[1] P. Turán, Eine Extremalaufgabe aus der Graphentheorie Mat. Fiz. Lapok 48 (1941) 436-452.

[2] A.A. Zykov, On some properties of linear complexes, Amer. Math. Soc. Transl. (1952) no. 79.