		Triv	Refl	Sgn
1	(1)	1	2	1
3	(12)	1	0	-1
2	(123)	1	-1	1

		(4)	(31)	(22)	(211)	(1^4)	G_x
1	(1)	1	3	2	3	1	24
6	(12)	1	1	0	-1	-1	4
3	(12)(34)	1	-1	2	-1	1	8
8	(123)	1	0	-1	0	1	3
6	(1234)	1	-1	0	1	-1	4

$$\langle \delta_x, \chi_\pi \rangle = \frac{1}{|G|} \sum_{y = gxg^{-1}} \overline{\chi_\pi(y)} = \frac{|G|/|G_x|}{|G|} \overline{\chi_\pi(x)}$$

$$= \frac{1}{|G_x|} \cdot \overline{\chi_\pi(x)}$$

So

$$|G_x| \; \delta_x(y) = \sum \overline{\chi_\pi(x)} \, \chi_\pi(y)$$

$$|G| \, \delta_1 = \sum \dim\pi \, \chi_\pi$$

<u>PLANCHEREL FORMULA:</u>

$$\cancel{f} \quad \delta_e = \sum \dim\pi \, \chi_\pi$$

IN GENERAL, $f \in C(G) \rightsquigarrow \pi(f)$

$$\pi(f)(v) := \int_G f(x) \, \pi(x) v \, dx$$

$$\chi_\pi(f) = \operatorname{tr} \pi(f).$$

<u>Rep's of S_4</u>: Triv, Sgn, Refl.

~~[illegible]~~ Sgn $\otimes$ Refl

$1 + 9 + ? + 9 + 1 = 24 \Rightarrow ? = 2.$

<u>Example</u>: $G = SU(2)$ $V(1/2)$ of dimension 2.

$V \otimes V = S^2 V \oplus \Lambda^2 V$ action of S_2

$V \otimes V \otimes V$ has an action of S_3

$$S^3 V + V_{refl} + \Lambda^3 V$$

Same for $G = S_4$ V 3-dim'l

$V \otimes V = S^2 V + \Lambda^2 V$

$3 \cdot 3 = \frac{3 \cdot 4}{2} + \frac{3 \cdot 2}{2}$

$6 + 3$

$$\chi_{S^2 V} + \chi_{\Lambda^2 V} = \chi_V^2$$

Character of $S^2 V$: ~~[illegible]~~

$\Lambda^2 V$:

$$\sum_{i,j}{}' \langle \pi(x) e_i \otimes \pi(x) e_j , e_j \otimes e_i \rangle = \text{~~[illegible]~~}$$

$$= \chi_{S^2 V} - \chi_{\Lambda^2 V}$$

$$\delta_{\hat{x}}(y) = \begin{cases} 1 & y \in O(x) \\ 0 & \text{otherwise} \end{cases}$$

$$\int_G \delta_{\hat{x}}(y) \cdot \overline{f(y)}\, dy =$$

$$\int_G \delta_{\hat{x}}(zyz^{-1})\, \overline{f(y)}\, dy = \int_G \int_G \delta_{\hat{x}}(zyz^{-1})\, \overline{f(y)}\, dy\, dz$$

Change the order of integration & change variables!

$u = zyz^{-1}$

$y = z^{-1}uz$

$$\int_G \delta_{\hat{x}}(u) \underbrace{\int_G \overline{f(z^{-1}uz)}\, dz}_{\text{class function}}\, du$$

$$Af(x) := \int_G f(gxg^{-1})\, dg$$

$A^2 = A$ called a projector:

$A^2 = A$

$A + (I-A) = I$

$(I-A)\cdot A = A(I-A) = 0$

$(I-A)^2 = 0$

So "Fourier Coefficients" only $\neq 0$ for class functions.

Basis χ_π.

$$\int_G \delta_{\hat{x}}(g)\overline{\chi_\pi(g)}\,dg =$$

$$= \frac{1}{|G|}\sum_{g\in O(\hat{x})} \overline{\chi_\pi(g)} = \cancel{|O(x)| \cdot \chi_\pi(\hat{x})}\ \frac{|O(x)|}{|G|}\overline{\chi_\pi(\hat{x})}$$

$$= \frac{1}{|G_x|}\overline{\chi_\pi(\hat{x})}.$$

Symplectic Groups:

<u>Φ:</u> Any bilinear nondegenerate symplectic form $\sim \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

<u>Pf:</u> Pick $e\neq 0$ ∄ $\Omega(e,e)=0$. $\exists f$ such that $\Omega(e,f)=1$; otherwise $\Omega(e,\cdot)=0\ \forall$.

$\Rightarrow e=0$. $\quad e=e_1 \quad f=f_1$

$$V^1 := \{v\in V : \Omega(e,v)=\Omega(f,v)=0\}$$

<u>claim:</u> $\Omega|_{V^1}$ is symplectic nondegenerate

clear.

Continue with $e_1 \ f_1 \ e_2 \ f_2 \cdots e_n \ f_n$

$\Rightarrow$ $\dim V$ is even. Order the basis $\{e_1 \cdots e_n \ f_1 \cdots f_n\}$

$\Omega \longleftrightarrow J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

$Sp(n, \mathbb{C}) \simeq \{ g \in GL(n, \mathbb{C}) : g^T J g = J \}$.

$sp(n, \mathbb{C}) \simeq \{ X \in gl(n, \mathbb{C}) : g^T J + J g = 0 \}$.

<u>Exercise:</u> $Sp(n, \mathbb{C})$ is path connected.

$$g \in Sp(n, \mathbb{C}) \Longrightarrow g^T \in Sp(n, \mathbb{C})$$
$$\bar{g} \in Sp(n, \mathbb{C})$$

$$J g^T J = -g^{-1}.$$

$Sp(n) := Sp(2n, \mathbb{C}) \cap U(2n) = \{ g \ni g^* g = I \ \& \ g^T J g = J \}$

<u>Exercise:</u> $g \in Sp(2n, \mathbb{C}) \Longrightarrow \det g = 1$

Follows from $(\det g)^2 = 1$ & path connected.

<u>$n=1$:</u> $Sp(1)$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\cancel{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}}$

$\cancel{\alpha \ \ t\bar{\beta} \ \ \beta}$ $\quad \cancel{\begin{pmatrix} a^* & c^* \\ b^* & d \end{pmatrix}}$ $\quad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$Sp(1) = SU(2) \simeq \cancel{Q} \, Q^\times$ $\quad \begin{pmatrix} -c & a \\ -d & b \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \cancel{\begin{pmatrix} 0 & ad-bc \\ ad+bc & 0 \end{pmatrix}}$

~~[illegible]~~ $Sp(n)$ is different

Want to know:

(1) what are the connected compact gps?

(2) When do Lie algebra homomorphisms exponentiate?

DEF: $\mathfrak{g}$ is called simple if it has no nontrivial ideals

G Compact $\Longrightarrow \mathcal{L}(G)$ has a positive/negative definite invariant inner product

Pf: integrate over G

$(\mathrm{Ad}g\,X, \mathrm{Ad}g\,Y) = (X, Y)$

For a matrix group $\mathrm{Ad}g\,X := gXg^{-1}$.

$\mathfrak{sl}(2,\mathbb{R})$ does not admit such an object

$$(gXg^{-1}, gYg^{-1}) = (X, Y)$$

$$\Longleftrightarrow ([Z,X],Y) + (X,[Z,Y]) = 0$$

$([H,E],E) + (E,[H,E]) = 0 \Longleftrightarrow (E,E) = 0$

$\mathfrak{su}(2)$ is different.

Symplectic Groups:

$\Omega / \mathbb{C}$ or $\mathbb{R}$ $\qquad \Omega(x,y) = -\Omega(y,x)$ &

Ω nondegenerate. $\Omega(x,y)=0 \ \forall y \Rightarrow x=0$

In coordinates: $(x,y) := \sum x_i y_i$

$\Omega(x,y) = (Jx, y)$ with $J^T = -J$ nondegenerate

Change of basis $J \to M^T J M$

Prop: $\exists$ basis $\ni$ $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Pf: ...

$Sp(2n, \mathbb{F}) = \{ g \in GL(n, \mathbb{F}) : g^T \cdot J g = J \}$

Exercise: $\det g = 1$ & $Sp(2n, \mathbb{F})$ is connected

$n=1$: $Sp(2) \simeq SL(2, \mathbb{C})$.

$$Q^\times = \{ x_0 + i x_1 + j x_2 + k x_3 : \sum x_i^2 = 1 \} \simeq SU(2)$$

$$x_0 + i x_1 + j x_2 + k x_3 \Longleftrightarrow \begin{pmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{pmatrix}$$

Real Forms: $\sigma : \mathfrak{g} \to \mathfrak{g} \ni$

(1) $\sigma([X,Y]) = [\sigma(X), \sigma(Y)]$

(2) $\sigma(X+Y) = \sigma(X) + \sigma(Y)$

(3) $\sigma(\alpha X) = \bar{\alpha} X$

(4) $\sigma^2 = Id$ & σ is 1-1

Lemma: $\mathfrak{g}^\sigma := \{X \in \mathfrak{g} : \sigma(X) = X\}$ is a real Lie algebra & $\mathfrak{g}^\sigma + \sqrt{-1}\,\mathfrak{g}^\sigma = \mathfrak{g}$.

Examples: $\mathfrak{g} = sp(2n, \mathbb{C})$.

(1) $\sigma(X) = \bar{X}$ $\qquad \mathfrak{g}^\sigma = sp(2n, \mathbb{R})$.

(2) $\sigma(X) = -\bar{X}^T$ $\qquad \mathfrak{g}^\sigma = sp(n)$

The corresponding group is compact.

NOTE: $-\bar{X}^T = J \bar{X} J^{-1}$ because

$J X^T = -XJ \qquad J^{-1} = -J$

FOR THE GROUPS $\bar{\ }$ makes sense.

The other classical Lie/matrix groups are similar:

$gl(n, \mathbb{C})$ $\quad \sigma(X) = \bar{X}$ gives $gl(n, \mathbb{R})$

$\sigma(X) = -\bar{X}^T$ gives $u(n)$.

same for $sl(n, \mathbb{C})$

$so(n, \mathbb{C})$ $\quad \sigma(X) = J_{p,q} \bar{X} J_{p,q}^{-1}$ $\quad \left.\begin{array}{l} J^T = J \\ J^2 = I \end{array}\right\}$

$$J = \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left(\begin{array}{c|c} I & \\ \hline & -I \end{array}\right) \end{array} \begin{array}{c} p \\ q \end{array}$$

Example: $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad so(2,1)$

Classification: Killing Cartan ... $\underbrace{A, B, C, D, E, F, G}_{6,7,8,4 \;\; 2}$

Find all "real" forms up to conjugacy

Example $\quad gl(2n, \mathbb{C})$ $\quad \sigma(X) = J \bar{X} J^{-1}$ $\quad u^*(2n)$

$\sigma(X) = J_{p,q} X^* J_{p,q}$ gives $su(p,q)$.

EXAMPLE: $G = SL(2,\mathbb{R})$

Proposition: Every $g \in G$ can be written uniquely $g = r(\theta)\cdot e^{X}$ where $X = X^T$ & $\mathrm{tr}\, X = 0$

(POLAR DECOMPOSITION)

Pf: $g^T g \in SL(2,\mathbb{R})$ is symmetric positive definite. $\exists\, x \in SL(2,\mathbb{R}) \ni$

$x(g^T g)x^{-1}$ is diagonal $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ $\alpha > 0$

Let $s = x^{-1}\begin{pmatrix} \alpha^{1/2} & 0 \\ 0 & \alpha^{-1/2} \end{pmatrix}x$. Then $s = e^{X}$ as "prescribed," and $s^2 = g^T g$.

Look at gs^{-1} $\quad (gs^{-1})(gs^{-1})^T = g s^{-2} g^T =$

$= g g^{-1} g^{T^{-1}} \cdot g^T = I$. So $gs^{-1} = r(\theta)$ and

$g = rs = re^{X}$.

Unique: ~~$r(\theta_1)\, s_1 = r(\theta_2)\cdot s_2 \in r(\theta) s_1 = s_2$~~

$g = r(\theta)s \implies s^2 = g^T \cdot g \implies$ only one such s

Corollary: $SL(2,\mathbb{R})$ is homotopic to $SO(2)$

$SO(2) = \{ r(\theta) : \theta \in \mathbb{R} \}$

What is $\widetilde{SL(2,\mathbb{R})}$?

Recall $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts on

$\mathcal{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$

$$g \cdot z = \frac{az+b}{cz+d} \qquad \operatorname{Im}(gz) = \frac{\operatorname{Im} z}{|cz+d|^2}$$

Define $j(g,z) = cz+d$. Then

$$j(g_1 g_2, z) = j(g_1, g_2 z) \cdot j(g_2, z).$$

(called a cocycle)

$$\widetilde{SL(2,\mathbb{R})} = \{ (g, \varphi_g(z)) : e^{\varphi_g(z)} = j(g,z) \}$$

$\varphi_g : \mathcal{H} \to \mathbb{C}$ cx analytic

Oscillator Representation

$\mathcal{S}(\mathbb{R})$ Schwartz space

$\Omega : sl(2) \longrightarrow \mathrm{End}(\mathcal{S}(\mathbb{R}))$

$\Omega(H) = x\partial_x + \frac{1}{2}$

$\Omega(E) = \frac{i}{2} m_{x^2}$

$\Omega(F) = \frac{i}{2}\partial_x^2$

THM: This rep'n exponentiates to a unitary rep'n of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R})$

$\Omega(e^{tH})f(x) = e^{t/2} f(e^t x)$

$\Omega(e^{tE})f(x) = e^{itx^2/2} f(x)$

$\Omega(e^{tF}) f(x) =$ convolution with $\frac{1+i}{2}\frac{1}{\sqrt{\pi t}} e^{-ix^2/2t}$

$\Omega(k)$

$2i\,\Omega(F-E) = x^2 - \partial_x^2$ $\qquad k = 2i(F-E)$

$2\Omega(k)$

Basis of Eigenvectors for $\Omega(k)$.

$a = x + \partial_x \qquad a^+ = x - \partial_x$

$(a^+)^* = a$ via $(f, g) := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$

$[a, a^+] = 2$

$[a, (a^+)^j] = 2j\,(a^+)^{j-1}$

$v_0 := e^{-\frac{x^2}{2}} \qquad a v_0 = 0$

$v_j = (a^+)^j v_0$ then

$a v_j = 2j\, v_{j-1}$

$(v_j, v_\ell) = 2^\ell \ell!\, \delta_{j\ell}(v_0, v_0) = 2^\ell \ell!\sqrt{\pi}\,\delta_{j\ell}$

$v_j = P_j(x)\, e^{-\frac{x^2}{2}}$

where P_j is a polynomial

$(-1)^j e^{x^2}\left(\frac{\partial}{\partial x}\right)^j (e^{-x^2})$

Hermite functions form an o.n. basis of $L^2(\mathbb{R})$.

x (means m_x) $= \frac{1}{2}(a+a^+)$

$\partial x = \frac{1}{2}(a-a^-)$

$\Omega(E,F) = \frac{i}{8}(a \pm a^+)^2$

$\Omega(k) = i\Omega(\cancel{F}-E) = \frac{1}{4}(aa^+ + a^+a)$.

$F-E$

$\Omega(E) = \frac{i}{8}(a+a^+)^2$ $\Omega(F) = \frac{i}{8}(a-a^+)$

$x = \frac{1}{2}(a+a^+)$ $\partial_x = \frac{1}{2}(a-a^+)$

$$[i(E-F),\ iH+(E+F)]$$

$$\cancel{\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$$

$$-[E-F, H] + i[E-F, E+F]$$

$$-(-2E-2F) + i[+H+H] = 2iH + 2(E+F)$$

$$[i(E-F),\ iH-(E+F)] = 2(E+F) - 2iH$$

$$=$$

$$\{iH+(E+F),\ i(E-F),\ iH-(E+F)\}$$

is another standard $\mathfrak{sl}(2,\mathbb{R})$.

$$\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\quad \begin{pmatrix}i & 1\\ 1 & -i\end{pmatrix},\quad \begin{pmatrix}i & -1\\ 1 & -i\end{pmatrix}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$h \qquad\qquad f \qquad\qquad e$$

$$(g_1, \varphi_{g_1}) \cdot (g_2, \varphi_{g_2}) = (g_1 g_2, \varphi_{g_1}(g_2 z) \cdot \varphi_{g_2})$$

$$e^{\varphi_{g_1}(g_2 z)} \cdot e^{\varphi_{g_2}(z)} = j(g_1, g_2 z) \cdot j(g_2, z)$$

$$= j(g_1 g_2, z) .$$

Covers of orthogonal groups

EXAMPLE 1: $G = SO(4,\mathbb{C})$

Let $\mathbb{C}^4 \simeq \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ as on the Exam

There is a nondegenerate bilinear form

$(X,Y) := tr(X J Y^T J)$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$J Y^T J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -y_4 & y_2 \\ y_3 & -y_1 \end{pmatrix}$$

So $(X,Y) = (-x_1 y_4 + x_2 y_3 + x_3 y_2 - x_4 y_1)$

$$(X,Y) \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

$SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ acts on $\mathbb{C}^4$ preserving $(\ ,\ )$.

So there is a map $\Psi : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to O(4,\mathbb{C})$

By continuity, it maps to $SO(4,\mathbb{C})$

$(g_1, g_2)\cdot X := g_1 X g_2^T$

WE NEED: $g^T J g = J \iff J g = (g^T)^{-1} J,\ J g^T = g^{-1} J$

CHECK: $\operatorname{tr}\left(g_1 X g_2^T \cdot J \cdot (g_1 \cdot Y \cdot g_2^T)^T \cdot J\right) =$

$= \operatorname{tr}\left(g_1 X \underbrace{g_2^T \cdot J \cdot g_2} \cdot Y^T \cdot g_1^T \cdot J\right) = \operatorname{tr}\left(g_1 X J Y^T \cdot g_1^T \cdot J\right)$

$= \operatorname{tr}\left(X \cdot J \cdot Y^T \cdot g_1^T \cdot J \cdot g_1\right) = \operatorname{tr}\left(X \cdot J \cdot Y^T \cdot J\right)$

FACT: $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ as bilinear forms

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boxtimes$$

Also $\sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right) \qquad \begin{array}{l} B + B^T = 0 \\ C + C^T = 0 \end{array}$$

Lie Algebra

$\dim \mathfrak{so}(4, \mathbb{C}) = 6 = \dim \mathfrak{sl}(2, \mathbb{C}) + \dim \mathfrak{sl}(2, \mathbb{C})$

FACTS:
- kernel of Ψ is $\pm(I, I)$.
- map is onto

Compute the differential at (I, I) and use the fact that a connected group is generated by a neighborhood of the identity

EXAMPLE 2: Specialize to the real form $SO(3,1)$.

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \simeq \mathbb{R}^4 \subseteq \mathbb{C}^4$$

These are the hermitian matrices.

$$\langle X, Y\rangle := tr(X \cdot J \cdot Y^* \cdot J)$$

$$tr \begin{pmatrix} t_1+x_1 & y_1+iz_1 \\ y_1-iz_1 & t_1-x_1 \end{pmatrix} \cdot \begin{pmatrix} -t_2+x_2 & y_2+iz_2 \\ y_2-iz_2 & -t_2-x_2 \end{pmatrix} =$$

$$= (t_1+x_1)(-t_2+x_2) + (y_1+iz_1)(y_2-iz_2)$$
$$+ (t_1-x_1)(-t_2-x_2) + (y_1-iz_1)(y_2+iz_2)$$

Set $X=Y$: $-2t_1^2 + 2x_1^2 + 2y_1^2 + 2z^2$

The orthogonal group is $O(1,3)$

Let $SL(2,\mathbb{C}) \hookrightarrow SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

$$g \longmapsto (g, \bar{g})$$

If $X^* = X$, then $(g,\bar{g})\cdot X = gXg^*$.

So $(gXg^*)^* = gX^*g^* = gXg^*$ preserves $\mathbb{R}^4$

Get a map $\varphi: SL(2,\mathbb{C}) \longrightarrow O(1,3)$.

FROM EARLIER, IMAGE IS IN $SO(1,3)$.

Image is connected; $\Psi(SL(2,\mathbb{C})) = SO(1,3)_0$.

Topology of $SO(1,3)$:

Cartan decomposition: Any $g \in SO(1,3)$ can be written uniquely as $g = u \cdot h$ where $u \in S[O(1) \times O(3)]$, $h = e^H$ with $H \in so(3,1)$ symmetric

"Pf" $g \in SO(1,3) \Longrightarrow g^T \in SO(1,3)$.

$g^T g = A$ as in the proof of the polar decomposition.

NEED: $A \in SO(1,3)$ symmetric

$$A \cdot I_{1,3} \cdot A = I_{1,3} \overset{???}{\Longrightarrow} \exists\, h \in SO(1,3) \ni h^2 = A.$$

If $A \cdot I_{1,3} \cdot A = I_{1,3}$ then $p(A) \cdot I_{1,3} \cdot p(A) = I_{1,3}$ for any polynomial $p(t)$.

$x \cdot A \cdot x^{-1} = \mathrm{diag}(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4)$ with $\lambda_j > 0$.

$\exists\ P_n(t) \longrightarrow \sqrt{t}$ uniformly on an interval containing λ_j. $P_n(A) \longrightarrow h' \ni h'^2 = A$.

$P_n(x^{-1} h' x) \longrightarrow x^{-1} h' x := h$ such that $h^2 = A$

$\therefore\ h \in SO(1,3)$.