G a compact (finite) group, (π, V) an irreducible representation.

Choose a basis $e_1, \dots, e_N$ such that $\pi(g)$ is unitary, $\pi(g)^* = \overline{\pi(g)}^T = \pi(g^{-1})$.

ASSUME: $FS\chi_\pi = 1$ (Frobenius-Schur Indicator)

$$FS(\pi) := \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g^2) = 1.$$

THEOREM: (π, V) has a basis $v_1, \dots, v_N$ such that $\pi(g)$ has real entries only

$\Longleftrightarrow$ $FS(\pi) = 1$.

"$\Longrightarrow$"

Proof: If a real basis exists, $\pi(g) = \overline{\pi(g)}$, ~~but~~ and $\pi(g)$ is an orthogonal matrix (with real entries). The bilinear form given by the inner product is invariant; means the trivial representation occurs in $S^2(V)$.

$\Longrightarrow$ $FS(\pi) = 1$.

Also $\operatorname{tr} \pi(g^2) = \sum \langle \pi(g^2) e_i, e_i \rangle =$

$= \sum \langle \pi(g) e_i, \pi(g)^T e_i \rangle$ & you can use

the Schur Orthogonality relation to get 1.

BUT THIS is "CLUMSY".

"$\Longleftarrow$" assume ~~%~~ $FS(\pi) = 1$.

(1)

Let A be any symmetric matrix. ~~Kohlac~~ Form

$$S_A = \frac{1}{|G|} \sum_{g \in G} \pi(g) \cdot A \cdot \pi(g)^T$$

Then S_A is symmetric and satisfies

$$\pi(g) \cdot S_A \cdot \pi(g)^T = S_A \quad \forall g \in G$$

This gives a bilinear form in $S^2(V)$.

$FS(\pi) = 1 \implies$ there is A such that $S_A \neq 0$

in which case it is also nondegenerate

(2) Consider $\overline{S}_A \cdot S_A$. Then

$\ast$ $\overline{S}_A \cdot S_A \cdot \pi(g) = \overline{S}_A \cdot (\pi(g)^T)^{-1} \cdot S_A =$

$$= \overline{S_A} \cdot \overline{\pi(g)} \cdot S_A = \overline{S_A \cdot \pi(g)} \cdot S_A =$$

$$= \overline{\pi(g^{-1})^T \circ S_A} \cdot S_A = \pi(g) \cdot \overline{S_A} \cdot S_A .$$

$S_A^* = \overline{S_A} \Rightarrow \overline{S_A} \cdot S_A = S_A^* \cdot S_A$

so

$\overline{S_A} \cdot S_A >> 0$

So $\overline{S_A} \cdot S_A = \lambda I$ with $\lambda \neq 0$; in fact $\lambda > 0$.

So by dividing by $\frac{1}{\sqrt{\lambda}}$ we get an ~~$\mathbb{R}$~~ invertible symmetric S such that $S^* \cdot S = I$.

Let $\{e_1 \dots e_N \quad ie_1 \dots ie_N\}$ be a real basis for V.

$$S \longleftrightarrow \begin{bmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{bmatrix} \qquad \overline{S} \longleftrightarrow \begin{bmatrix} S_1 & -S_2 \\ S_2 & S_1 \end{bmatrix}$$

and S_1, S_2 are (real) symmetric

$\overline{S} \cdot S = I \iff$ ~~$A_1 A_2 = A_2 A_1$ and~~ $S_1 S_2 = S_2 S_1$ and $S_1^2 + S_2^2 = I$

$= S \cdot \overline{S}$

same as $\begin{bmatrix} S_1 & S_2 \\ +S_2 & -S_1 \end{bmatrix} \cdot \begin{bmatrix} S_1 & S_2 \\ +S_2 & -S_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

The $+1$ eigenspace of

$a = \begin{bmatrix} S_1 & S_2 \\ S_2 & -S_1 \end{bmatrix}$ has dimension n, and forms a basis of $V/\mathbb{C}$.

THIS IS THE BASIS WE WANT