Covers of orthogonal groups

EXAMPLE 1: $G = SO(4,\mathbb{C})$

Let $\mathbb{C}^4 \simeq \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ as on the Exam

There is a nondegenerate bilinear form

$(X,Y) := tr(X J Y^T J)$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$J Y^T J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -y_4 & y_2 \\ y_3 & -y_1 \end{pmatrix}$$

So $(X,Y) = (-x_1y_4 + x_2y_3 + x_3y_2 - x_4y_1)$

$$(X,Y) \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

$SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ acts on $\mathbb{C}^4$ preserving $(\ ,\ )$.

So there is a map $\Psi : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to O(4,\mathbb{C}$

By continuity, it maps to $SO(4,\mathbb{C})$

$(g_1, g_2)\cdot X := g_1 X g_2^T$

WE NEED: $g^T J g = J \iff Jg = (g^T)^{-1} J,\ Jg^T = g^{-1} J$

CHECK: $tr\left(g_1 X g_2^T \cdot J \cdot (g_1 \cdot Y \cdot g_2^T)^T \cdot J \right) =$

$= tr\left(g_1 X \underbrace{g_2^T \cdot J \cdot g_2} \cdot Y^T \cdot g_1^T \cdot J \right) = tr\left(g_1 X J Y^T \cdot g_1^T \cdot J \right)$

$= tr\left(X \cdot J \cdot Y^T \cdot g_1^T \cdot J \cdot g_1 \right) = tr\left(X \cdot J \cdot Y^T \cdot J \right)$

FACT: $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ as bilinear forms

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boxtimes$$

Also $\sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array} \right) \qquad \begin{array}{l} B + B^T = 0 \\ C + C^T = 0 \end{array}$$

Lie Algebra

$\dim so(4, \mathbb{C}) = 6 = \dim sl(2, \mathbb{C}) + \dim sl(2, \mathbb{C})$

FACTS: • kernel of Ψ is $\pm(I, I)$.
• map is onto

Compute the differential at (I, I) and use the fact that a connected group is generated by a neighborhood of the identity.

EXAMPLE 2: Specialize to the real form $SO(3,1)$.

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \simeq \mathbb{R}^4 \subseteq \mathbb{C}^4$$

These are the hermitian matrices.

$$\langle X, Y\rangle := tr(X \cdot J \cdot Y^* \cdot J)$$

$$tr\begin{pmatrix} t_1+x_1 & y_1+iz_1 \\ y_1-iz_1 & t_1-x_1 \end{pmatrix} \cdot \begin{pmatrix} -t_2+x_2 & y_2+iz_2 \\ y_2-iz_2 & -t_2-x_2 \end{pmatrix} =$$

$$= (t_1+x_1)(-t_2+x_2) + (y_1+iz_1)(y_2-iz_2)$$
$$+ (t_1-x_1)(-t_2-x_2) + (y_1-iz_1)(y_2+iz_2)$$

Set $X=Y$: $-2t_1^2 + 2x_1^2 + 2y_1^2 + 2z^2$

The orthogonal group is $O(1,3)$

Let $SL(2,\mathbb{C}) \hookrightarrow SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

$$g \longmapsto (g, \bar{g})$$

If $X^* = X$, then $(g,\bar{g}) \cdot X = gXg^*$.

So $(gXg^*)^* = gX^*g^* = gXg^*$ preserves $\mathbb{R}^4$

Get a map $\Psi: SL(2,\mathbb{C}) \longrightarrow O(1,3)$.

FROM EARLIER, IMAGE IS IN $SO(1,3)$.

Image is connected; $\Psi(SL(2,\mathbb{C})) = SO(1,3)_0$.

Topology of $SO(1,3)$:

Cartan decomposition: Any $g \in SO(1,3)$ can be written uniquely as $g = u \cdot h$ where $u \in S[O(1) \times O(3)]$, $h = e^{H}$ with $H \in so(3,1)$ symmetric

"Pf" $g \in SO(1,3) \Longrightarrow g^T \in SO(1,3)$.

$g^T g = A$ as in the proof of the polar decomposition.

NEED: $A \in SO(1,3)$ symmetric

$$A \cdot I_{1,3} \cdot A = I_{1,3} \overset{???}{\Longrightarrow} \exists\, h \in SO(1,3) \ni h^2 = A.$$

$$A \cdot I_{13} \cdot A = I_{13} \iff A \cdot I_{13} = I_{13} \cdot A^{-1} \iff p(A) \cdot I_{13} = I_{13} \cdot p(A^{-1})$$

$x \cdot A \cdot x^{-1} = \operatorname{diag}(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4)$ with $\lambda_j > 0$.

$\exists\ P_n(t) \longrightarrow \sqrt{t}$ uniformly on an interval containing λ_j. $P_n(A) \longrightarrow h' \ni h'^2 = A$.

$P_n(x^{-1} h' x) \longrightarrow x^{-1} h' x := h$ such that $h^2 = A$

$\therefore\ h \in SO(1,3)$.