(π, V) a representation of a compact group

QUE: When does V have a basis such that $\pi(g)$ is real?

NECESSARY CONDITION: $\chi_\pi(g) = \operatorname{tr} \pi(g)$ takes real values only.

This is not SUFFICIENT.

For example: $G = SU(2)$.

$N = 2\ell$ $\quad g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

Eigenvalues are $\ell \;\; \ell-1 \; \dots \; -\ell$

$$\operatorname{tr} \pi_N(r(\theta)) = e^{i\ell\theta} + e^{i(\ell-1)\theta} + \cdots + e^{-i(\ell-1)\theta} + e^{-i\ell\theta}$$

~~But~~ This is real (sum of cosines) but not all rep's are real.

QUE: When is $\chi_\pi(g)$ real?

Choose an o.n. basis. Then

$$\pi(g)^* = \pi(g^{-1}) \Longleftrightarrow \overline{\pi(g)} = \pi({g^T}^{-1}).$$

Two representations:

$$(\pi, V) \quad \text{and} \quad (\overline{\pi}, V) \simeq (\pi^*, V^*).$$

dual NOT
hermitian dual

$$\chi_{\overline{\pi}}(g) = \overline{\chi_{\pi}(g)}$$

So $\operatorname{tr} \pi(g)$ real $\iff$ $\pi \simeq \pi^*$

MEANS: $\exists \; 0 \neq A \in \operatorname{Hom}_G(V, V^*)$

Linear Maps from V to V^* such that

$$A \circ \pi(g) = \pi^*(g) \circ A$$

$\operatorname{Hom}(V, V^*) \simeq V \otimes V$

Same as: There exists a nondegenerate invariant bilinear form.

$$A : V \longrightarrow V^* \longleftrightarrow \mathcal{B}_A(v_1, v_2) := (Av_1)(v_2).$$

Same as: The trivial representation occurs in $V \otimes V$.

FACT: If (π_1, V_1) & (π_2, V_2) are irreducible, the trivial repn occurs at most once

This is Schur's lemma in disguise.

<u>FACT:</u> $V \otimes V \simeq S^2(V) \oplus \Lambda^2(V)$

symmetric and antisymmetric forms.

The trivial representation ~~occurs~~ only

- once in $S^2(V)$
- once in $\Lambda^2(V)$
- never

So we need the characters

$\chi_{S^2(V)}$ and $\chi_{\Lambda^2 V}$

For a finite group:

$$\frac{1}{|G|} \sum \chi_{S^2 V}(g) = 1$$

or

$$\frac{1}{|G|} \sum \chi_{\Lambda^2 V}(g) = 1$$

or

both are 0.

$\chi_{V\otimes V}$: o.n. Basis is $e_i \otimes e_j$

$$\sum_{i,j} \langle \pi(g)e_i, e_i\rangle \cdot \langle \pi(g)e_j, e_j\rangle$$

$$= \chi_\pi(g)^2$$

χ_{S^2V} and $\chi_{\Lambda^2 V}$:

$\pi(g)$ is diagonalizable ; eigenvalues $\lambda_1 \ldots \lambda_N$

$$\operatorname{tr} \pi(g) = \sum \lambda_j$$

$(S^2\pi)(g)$ has eigenvalues $\lambda_i \lambda_j \quad i \leq j$

$(\Lambda^2\pi)(g)$ has eigenvalues $\lambda_i \lambda_j \quad i < j$

$S^2\pi \oplus \Lambda^2\pi = \pi \otimes \pi$ character $\chi_\pi(g)^2$

$S^2\pi - \Lambda^2\pi$ has "eigenvalues" $\lambda_i^2 \quad \chi_\pi(g^2)$.

$$\chi_{S^2\pi} = \frac{1}{2}\left[\chi_\pi(g)^2 + \chi_\pi(g^2)\right]$$

$$\chi_{\Lambda^2\pi} = \frac{1}{2}\left[\chi_\pi(g)^2 - \chi_\pi(g^2)\right]$$

$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g^2) = \begin{cases} 1 & \text{orthogonal} \\ 0 & \text{complex} \\ -1 & \text{symplectic (quaternionic)} \end{cases}$$

Examples:

$G = S^1$ except for the trivial representation, none have real characters.

$e^{in\theta} \oplus e^{-in\theta}$ is real irreducible $/\mathbb{R}$

reducible $/\mathbb{C}$

$G = S_4$: $(1)^2 = (1)_1$ $\quad (12)^2 = (1)_6$

$(12)(34)^2 = (1)_3 \quad (123)^2 = (123)(123) = (132)_8$

$(1234)^2 = (1234)(1234) = (13)(24)_3$

	(4)	(31)	(22)	(211)	(1^4)	
(1)	1	3	2	3	1	10
(12)(34)	1	−1	2	−1	+1	6
(123)	1	0	−1	0	1	8
	24	24	24	24	24	

NONE ARE SYMPLECTIC.

$\underline{G = SU(2)}$ odd dim'l orthogonal

even dim'l symplectic

→ see Bröcker–tomDieck ←

A finite Group:

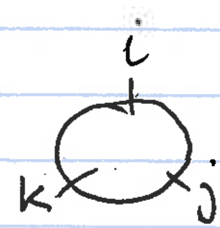
$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \hookrightarrow SU(2) \quad |Q_8| = 8$

$$\pm 1 \mapsto \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pm i \to \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\pm j \to \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\pm k \to \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$



Conjugacy classes: $\{-1\}$ $\{+1\}$, $\{i,-i\}$, $\{j,-j\}$, $\{k,-k\}$

$i \cdot i \cdot (-i) = i \qquad j \cdot i(-j) = -jk = -i, \quad ki(-k) = jk = i$

$i \cdot j(-i) = -ki = -j \qquad k \cdot j \cdot (-k) = -ik = j$

	~~triv~~	~~2-dim'l~~	~~ε~~	~~δ~~	~~M~~
~~1~~	~~1~~	~~2~~			
~~i~~	~~1~~	~~0~~			
~~j~~	~~1~~	~~0~~			
~~k~~	~~1~~	~~0~~			

		2 dim'l	
1	1	2	
1	-1	-2	
2	$\pm i$	0	
2	$\pm j$	0	
2	$\pm k$	0	

<u>Squares</u>: $2\ \{1\}$ $\quad 2$

$6\ \{-1\}$ $\quad -2$

-8

<u>QUE</u>: What are the other ~~repres~~ irreducible representations?

They must all be 1 dimensional

$$1 \longrightarrow \{\pm 1\} \longrightarrow Q_8 \longrightarrow \mathbb{Z}_2^4 \longrightarrow 1$$